Covers of groups definable in o-minimal structures

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Abstract

We develop in this paper the theory of covers for Hausdorff properly $\bigvee$-definable manifolds with definable choice in an o-minimal structure $\mathcal{N}$. In particular, we show that given an $\mathcal{N}$-definably connected $\mathcal{N}$-definable group G we have $1 \to \pi_1(G) \to \widetilde{G} \xrightarrow{p} G \to 1$ in the category of strictly properly $\bigvee$-definable groups with strictly properly $\bigvee$-definable homomorphisms, where $\pi_1(G)$ is the o-minimal fundamental group of G.

1

1 Introduction

Throughout this paper, $\mathcal{N}$ will be an o-minimal structure and definable will mean $\mathcal{N}$-definable (possibly with parameters). We will assume the readers familiarity with the basic theory of o-minimal structures (see for example [vdd]).

In section 2, we will introduce several categories - properly $\bigvee$-definable manifolds with (properly) $\bigvee$-definable maps and properly $\bigvee$-definable manifolds with strictly (properly) $\bigvee$-definable maps - generalising the category of definable manifolds with definable maps. Properly $\bigvee$-definable subsets of properly $\bigvee$-definable manifolds, which will play an important role, are introduced and several notions such as properly $\bigvee$-definable cell decomposition, properly $\bigvee$-definable completeness, properly $\bigvee$-definable connectedness, dimensions and generic points are studied. These notions, generalise corresponding notions for definable subsets of definable manifolds and are shown to be invariant under the model theoretic operations of taking elementary extensions, elementary substructures, reducts and expansions.

In section 3, we introduce the o-minimal fundamental group functor for the category of properly $\bigvee$-definable manifolds with definable choice together with continuous strictly properly $\bigvee$-definable maps. The construction of this functor is based on k-cells ($k = 0, 1, 2$) with an orientation, and we prove that this functor satisfies all the relevant properties that one should expect for the fundamental group functor. Namely, the o-minimal Tietze and Seifert-van Kampen theorems are proved for properly $\bigvee$-definably complete, locally finite properly $\bigvee$-definable manifolds with definable choice, generalising results from [bo] on o-minimal fundamental groups of definable sets in o-minimal expansions of real closed fields. Moreover, the o-minimal fundamental group of a properly $\bigvee$-definably complete, properly $\bigvee$-definable manifold with definable choice is proved to be invariant under the model theoretic operations of taking elementary extensions, elementary substructures, reducts and expansions. In section 4, all the theory of strictly properly $\bigvee$-definable covering spaces is developed in the category of properly $\bigvee$-definable manifolds with definable choice together with continuous strictly properly $\bigvee$-definable maps.

Finally, in section 5 we apply our previous theory to $\bigvee$-definable groups - which we prefer to call in this paper, strictly properly $\bigvee$-definable groups mainly for two reasons: first because we do not assume that $\mathcal{N}$ is $\aleph_1$-saturated (as in [pst2], where part of the theory of $\bigvee$-definable groups is developed)

and secondly because, as we show throughout the paper, the distinction between properly $\bigvee$-definable subgroups and $\bigvee$-definable subgroups is a very important one, with a better theory in the first case. We start section 5 by developing the basic theory of strictly properly $\bigvee$-definable groups: existence of a properly $\bigvee$-definable manifold structure making the group operations and strictly properly $\bigvee$-definable homomorphisms continuous (this is already implicit in [pst2]); DCC for strictly properly $\bigvee$-definable subgroups; existence of infinite strictly properly $\bigvee$-definable abelian subgroups of infinite strictly properly $\bigvee$-definable groups; existence of strictly properly $\bigvee$-definable quotient of a strictly properly $\bigvee$-definable group by a strictly properly $\bigvee$-definable normal subgroup and existence of a corresponding strictly properly $\bigvee$-definable section; centerless strictly properly $\bigvee$-definable groups with no strictly properly $\bigvee$-definable normal subgroups of positive dimension are shown to be the $\bigvee$-definable open and closed subgroups of definably semisimple definable groups generated by open definable subsets and, the classification of solvable (and therefore by the above, of all) strictly properly $\bigvee$-definable groups is reduced to the classification of properly $\bigvee$-definably complete, strictly properly $\bigvee$-definable solvable groups; existence of definable choice is proved for strictly properly $\bigvee$-definable groups and the theory of strictly properly $\bigvee$-definable coverings of strictly properly $\bigvee$-definable groups is presented.

There are two problems that we do not handle in this paper. The first is the classification of properly $\bigvee$-definably complete, properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable solvable groups. We consider this problem in more detail in [e3]. The second problem is the computation of the o-minimal fundamental group of a definable group (or even more generally of a strictly properly $\bigvee$-definable group). We show here that such groups are abelian and finitely generated, and in [e2] where we compute the o-minimal fundamental groups of groups definable in o-minimal expansions of real closed fields, we reduce as well this problem to the problem of computing the o-minimal fundamental groups of definably compact, definable abelian groups. We show here that the o-minimal fundamental group $\pi_1(G)$ of a definably compact, definable abelian group G is a torsion-free finitely generated abelian group. We conjecture that the rank of $\pi_1(G)$ the equals the dimension of G. This problem however, can only be solved using a general cohomology theory for general o-minimal structures which we hope to develop in [e4].

2 Properly $\bigvee$-definable manifolds

2.1 Properly $\bigvee$-definable manifolds

Definition 2.1 A *properly $\bigvee$-definable manifold (over A) of dimension m* where $A \subseteq N$ is such that $|A| < \aleph_0$, is a triple $\mathbf{X} := (X, (X_i, \phi_i)_{i \in I})$ where $\{X_i : i \in I\}$ is cover of the set X with $|I| < \aleph_1$ such that for each $i \in I$, if $I_i := \{j \in I : X_i \cap X_j \neq \emptyset\}$ then we have injective maps $\phi_i : X_i \longrightarrow N^m$ such that $\phi_i(X_i)$ is an open definably connected definable set (over A), for all $j \in I_i$, $\phi_i(X_i \cap X_j)$ is an definable (over A) open subset of $\phi_i(X_i)$ and the map $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} := \phi_j \circ \phi_i^{-1}$ is a definable homeomorphism (over A).

A properly $\bigvee$-definable manifold (over A) of dimension m will be called a *locally finite properly $\bigvee$-definable manifold (over A) of dimension m* if $|I_i| < \aleph_0$; a properly $\bigvee$-definable manifold (over A) of dimension m will be called a *definable manifold (over A) of dimension m* if $|I| < \aleph_0$.

For the rest of the section, let $\mathbf{X} = (X, (X_i, \phi_i)_{i \in I})$ and $\mathbf{Y} = (Y, (Y_j, \psi_j)_{j \in J})$ be properly $\bigvee$-definable manifolds of dimension m and n respectively and defined over A_X and A_Y respectively. Note that, $\mathbf{X} \times \mathbf{Y} = (X \times Y, (X_i \times Y_j, (\phi_i, \psi_j))_{i \in I, j \in J})$ is then a properly $\bigvee$-definable manifold of dimension $m \times n$ and defined over $A_X \cup A_Y$.

Definition 2.2 Given $Z \subseteq X$ let $I^Z := \{i \in I : Z \cap X_i \neq \emptyset\}$ (necessarily $|I^Z| < \aleph_1$) and for each $i \in I^Z$, let $Z_i := Z \cap X_i$. Let $A_X \subseteq B \subseteq N$ be such that $|B| < \aleph_0$. We say that Z is a *properly $\bigvee$-definable subset of X (over B)* if for each $i \in I^Z$, the set $\phi_i(Z_i)$ is a definable subset of $\phi_i(X_i)$ (over B); we say that Z is a *definable subset of X (over B)* if Z is a properly $\bigvee$-definable subset of X (over B) and there is a finite subset I_0 of I such that $Z \subseteq \cup\{X_i : i \in I_0\}$. Finally, we say that Z is a *$\bigvee$-definable subset of X (over B)* if $Z = \cup\{Z^s : s \in S\}$ where $|S| < \aleph_1$ and for each $s \in S$, Z^s is is a definable subset of X (over B).

Of course, a definable subset of X is a properly $\bigvee$-definable subset of X, a properly $\bigvee$-definable subset of X is a $\bigvee$-definable subset of X and X is always a properly $\bigvee$-definable subset of X. By a definable (resp., properly $\bigvee$-definable and $\bigvee$-definable) subset U of a $\bigvee$-definable subset Z of X we mean a definable (resp., properly $\bigvee$-definable and $\bigvee$-definable) subset U of

X which is a subset of Z. Definable subsets of X have a very well behaved theory, induced from the theory of definable sets in N^m. The collection of properly $\bigvee$-definable subsets of X which is closed under finite unions, finite intersections and under taking complements, but not under the projection maps (in fact, general $\bigvee$-definable subsets typically occur in this way -see the examples below), will also have a well behaved theory as we shall show throughout the paper. However, general $\bigvee$-definable subsets do not have an interesting theory when $\mathcal{N}$ is not $\aleph_1$-saturated, consider for example in $\mathcal{N} = (\mathbb{R}, <)$ the $\bigvee$-definable subset $\mathbb{Q}$ of $\mathbb{R}$. Fortunately, general $\bigvee$-definable subsets will not play an essential role in the construction of o-minimal fundamental groups and covers, so we will work most of the time without the $\aleph_1$-saturation condition.

Definition 2.3 Let Z be a properly $\bigvee$-definable subset of X (over B). A map $f : Z \subseteq X \longrightarrow Y$ is a *properly $\bigvee$-definable map (over B)* if its graph $\Gamma(f)$ is a properly $\bigvee$-definable subset of $X \times Y$ (over B). Similarly, we say that a properly $\bigvee$-definable map $f : Z \longrightarrow Y$ is a *definable map (over B)* if its graph $\Gamma(f)$ is a definable subset of $X \times Y$ (over B). And finally if Z is a $\bigvee$-definable subset of X (over B) and $f : Z \longrightarrow Y$ is a map, we say that f is a *$\bigvee$-definable map (over B)* if $Z = \cup \{Z^s : s \in S\}$ with $|S| < \aleph_1$, where for each $s \in S$, Z^s is a definable subset of X (over B) and $f_{|Z^s}$ is a definable map (over B).

Note that: (1) $f : Z \subseteq X \longrightarrow Y$ is a properly $\bigvee$-definable map (over B) iff for all $i \in I^Z$, and for all $j \in J^{f(Z_i)}$, the map $\psi_j \circ f \circ \phi_{i|}^{-1} : \phi_i(Z_i) \longrightarrow \psi_j(Y_j)$ is a definable map (over B); (2) if $f : Z \longrightarrow Y$ is a properly $\bigvee$-definable map (over B) then the image $f(U)$ of a definable subset U of Z (over B) is not necessarily a definable subset of Y but a it is a properly $\bigvee$-definable subset of Y (over B) and therefore, the image $f(U)$ of a (properly) $\bigvee$-definable subset U of Z (over B) is in general a $\bigvee$-definable subset of Y (over B) (e.g., take $\mathcal{N} = (\mathbb{Q}, <)$, $I = \mathbb{N}$ and $X = \cup \{X_i : i \in I\}$ with $X_i = \{i\}$, $J = \{1\}$ and $Y = Y_1 = \mathbb{Q}$, and $f(n) = \frac{1}{n}$); (3) if $f : Z \longrightarrow Y$ is a properly $\bigvee$- definable map (over B) then the inverse image $f^{-1}(W)$ of a definable subset W of $f(Z)$ (over B) is not necessarily a definable subset of Z but a it is a properly $\bigvee$-definable subset of Z (over B) and therefore, the inverse image $f^{-1}(W)$ of a (properly) $\bigvee$-definable subset W of $f(Z)$ (over B) is in general a $\bigvee$-definable subset of Z (over B) (e.g., take $\mathcal{N} = (\mathbb{Q}, <)$, $I = \{1\}$, $X = X_1 = \mathbb{Q}^{\geq 0}$, $J = \mathbb{N}$

and $Y = \cup\{Y_j : j \in J\}$ with $Y_j = \{j\}$ and $f(x) = 2k$ if $x \in (\frac{k}{2}, \frac{k+1}{2})$ and $f(\frac{k}{2}) = 2k+1$, where $k \in \mathbb{N}$ and take $W = 2\mathbb{N}+1$) and (4) if $f : Z \subseteq X \longrightarrow Y$ is a properly $\bigvee$-definable map (over B) then, we have $Z = \cup\{Z^s : s \in S\}$ with $|S| < \aleph_1$, where for each $s \in S$, Z^s is a definable subset of X (over B) and $f_{|Z^s}$ is a definable map (over B).

Definition 2.4 Let Z be a properly $\bigvee$-definable subset of X (over B). A properly $\bigvee$-definable map $f : Z \subseteq X \longrightarrow Y$ (over B) is a *strictly properly $\bigvee$-definable map (over B)* if for all $i \in I^Z$, $f(Z_i)$ is a definable subset of Y. Finally, any $\bigvee$-definable map $f : Z \subseteq X \longrightarrow Y$ (over B) is a *strictly $\bigvee$-definable map (over B)* since we can write $Z = \cup\{Z^s : s \in S\}$ with $|S| < \aleph_1$, where for each $s \in S$, Z^s is a definable subset of X (over B) and $f_{|Z^s}$ is a definable map (over B) such that $f(Z_s)$ is a definable subset of Y.

X can be made into a topological space: the basis for the topology is the collection of *open definable subsets of X* i.e., definable subsets U of X such that for all $i \in I^U$, $\phi_i(U_i)$ is an open definable subset of $\phi_i(X_i)$. We will often identify two properly $\bigvee$-definable manifolds **X** and **Y** if $X = Y$ and the identity map $1_X : X \longrightarrow Y$ is a strictly (properly) $\bigvee$-definable homeomorphism. **Y** is a *strictly (properly) $\bigvee$-definable submanifold* of **X** if Y is a (properly) $\bigvee$-definable subset of X, and the inclusion map $Y \longrightarrow Y \subseteq X$ is a strictly (properly) $\bigvee$-definable homeomorphism onto its image. The strictly (properly) $\bigvee$-definable submanifolds **Y** of N^n are called *strictly (properly) $\bigvee$-definable affine manifolds*.

2.2   Properly $\bigvee$-definable cell decomposition

In this subsection **X** will be a locally finite properly $\bigvee$-definable manifold. There are many geometric properties of definable sets and definable maps in the o-minimal structure $\mathcal{N}$. However, two of the most powerful results that we will be using throughout this paper are the monotonicity theorem for definable one variable functions and the C^p-cell decomposition theorem for definable sets and definable maps. We will now explain the C^p-cell decomposition theorem (here $p = 0$ if $\mathcal{N}$ is not an expansion of a (real closed) field) in order to introduce as well the notions of properly $\bigvee$-definable C^p-cell decomposition and o-minimal dimension of properly $\bigvee$-definable subsets of X.

Definition 2.5 C^p-cells and o-minimal dimension are defined inductively as follows: (i) the unique non empty definable subset of N^0 is a C^p-cell of dimension zero, a point in N^1 is a C^p-cell of dimension zero and an open interval in N^1 is a C^p-cell of dimension one; (ii) a C^p-cell in N^{l+1} of dimension k (resp., $k+1$) is a definable set of the form $\Gamma(f)$ (the graph of f) where $f : C \longrightarrow N$ is a C^p-definable function and C is a C^p-cell in N^l of dimension k (resp., of the form $(f,g)_C := \{(x,y) \in C \times N : f(x) < y < g(x)\}$ where $f, g : C \longrightarrow N$ are definable C^p-function with $-\infty \le f < g \le +\infty$ and C is a C^p-cell in N^l of dimension k. The Euler characteristic $E(C)$ of a C^p-cell C of dimension k is defined to be $(-1)^k$.

Definition 2.6 A C^p-cell decomposition of N^m is a special kind of partition of N^m into finitely many C^p-cells: a partition of N^1 into finitely many disjoint C^p-cells of dimension zero and one is a C^p-cell decomposition of N^m and, a partition of N^{k+1} into finitely many disjoint C^p-cells $C_1, \ldots, C_m$ is a C^p-cell decomposition of N^{k+1} if $\pi(C_1), \ldots, \pi(C_m)$ is a C^p-cell decomposition of N^k (where $\pi : N^{k+1} \longrightarrow N^k$ is the projection map onto the first k coordinates). Let $A_1, \ldots A_k \subseteq A \subseteq N^m$ be definable sets. A C^p-cell decomposition of A compatible with $A_1, \ldots A_k$ is a finite collection $C_1, \ldots, C_l$ of C^p partitioning A obtained from a C^p-cell decomposition of N^m such that for every $(i,j) \in \{1, \ldots, k\} \times \{1, \ldots, l\}$ if $C_j \cap A_i \ne \emptyset$ then $C_j \subseteq A_i$.

Fact 2.7 *[vdd] Given definable sets $A_1, \ldots, A_k \subseteq A \subseteq N^m$ there is a C^p-cell decomposition of A compatible with $A_1, \ldots A_k$ and, for every definable function $f : A \longrightarrow N$, $A \subseteq N^m$, there is a C^p-cell decomposition of A, such that each restriction $f_{|C} : C \longrightarrow N$ is C^p for each cell $C \subseteq A$ of the C^p-cell decomposition.*

The o-minimal dimension $dim(A)$ and Euler characteristic $E(A)$ of a definable set A are defined by $dim(A) = \max\{dim(C) : C \in \mathcal{C}\}$ and $E(A) = \sum_{C \in \mathcal{C}} E(C)$ where $\mathcal{C}$ is some (equivalently any) C^p-cell decomposition of A. These notions are well behaved under the usual set theoretic operations on definable sets, are invariant under definable bijections and given a definable family of definable sets, the set of parameters whose fibre in the family has a fixed dimension (resp., Euler characteristic) is also a definable set. The cell decomposition theorem is also used to show that every

definable set has only finitely many definably connected components, and given a definable family of definable sets there is a uniform bound on the number of definably connected components of the fibres in the family.

Remark 2.8 Let $A, B \subseteq N^m$ be definable sets and $\phi : A \longrightarrow B$ a definable homeomorphism. If $C_1, \ldots, C_n$ is a cell decomposition of A then, $\phi(C_1), \ldots, \phi(C_n)$ is a cell decomposition of B.

Definition 2.9 Let $A_1, \ldots, A_n, B, Z$ be properly $\bigvee$-definable subsets of X. Let $I = \{1, 2, \ldots\}$ be an enumeration of I. Define inductively (X'_i, M_i, N_i) for $i \in I$ by: $X'_1 = X_1$, M_1 is a cell decomposition of $\phi_1(X_1)$ compatible with the definable subsets $\phi_1(X_1 \cap X_j)$, $\phi_1(X_1 \cap A_l)$ and $\phi_1(X_1 \cap X_j \cap A_l)$ for all $l \in \{1, \ldots, n\}$ and all $j \in I_1$, and $N_1 = M_1$ (recall that for $i \in I$, $I_i := \{j \in I : X_i \cap X_j \neq \emptyset\}$ is finite); let $X'_{i+1} := X_{i+1} \setminus \cup \{C : \exists r \in \{1, \ldots, i\}, \ \phi_r(C) \in N_i\}$ and M_{i+1} is a cell decomposition of $\phi_{i+1}(X'_{i+1})$ compatible with the definable sets $\phi_{i+1}(X'_{i+1} \cap X_j)$, $\phi_{i+1}(X'_{i+1} \cap A_l)$ and $\phi_{i+1}(X'_{i+1} \cap X_j \cap A_l)$ for all $l \in \{1, \ldots, n\}$ and all $j \in I_{i+1}$ and such that N_{i+1} which is equal to M_{i+1} together with all the cell in N_j for $j \in I_{i+1} \cap \{1, \ldots, i\}$ is a cell decomposition of $\phi_{i+1}(X_{i+1})$.

We define a properly $\bigvee$-definable cell decomposition of X compatible with $A_1, \ldots, A_n$ to be a sequence $K = \{C : \phi_i(C) \in N_i \ for \ some \ i \in I\}$ some $\{N_i : i \in I\}$ like above. By a properly $\bigvee$-definable cell decomposition of B compatible with $A_1, \ldots, A_n$ we mean a sequence $K_B = \{C \subseteq B : C \in K\}$ for some properly $\bigvee$-definable cell decomposition K of X compatible with $A_1, \ldots, A_n, B$. Note that if $C \in K$ then, for all $j \in I$ if $C \cap X_j \neq \emptyset$ then $C \subseteq X_j$ and $\phi_j(C)$ is a k-cell in $\phi_j(X_j)$. If $C \subseteq Z$ we say that C is a k-cell (of K) in Z.

From fact 2.7 and definition 2.9 we get:

Fact 2.10 *Given properly $\bigvee$-definable subsets $A_1, \ldots, A_k \subseteq A \subseteq X$ there is a C^p-properly $\bigvee$-definable cell decomposition of A compatible with $A_1, \ldots A_k$ and, for every strictly properly $\bigvee$-definable map $f : A \longrightarrow N$, there is a C^p-properly $\bigvee$-definable cell decomposition of A, such that each restriction $f_{|C} : C \longrightarrow N$ is C^p for each cell $C \subseteq A$ of the C^p-properly $\bigvee$-definable cell decomposition.*

There is no $\bigvee$-definable cell decomposition of general $\bigvee$-definable subsets of X and there is no corresponding $\bigvee$-definable cell decomposition theorem.

If $\mathbf{X}$ is a properly $\bigvee$-definable but not locally finite properly $\bigvee$-definable manifold, then there is no cell decomposition theorem for general (properly) $\bigvee$-definable subsets of X, however if Z is a properly $\bigvee$-definable subset of X for which there is a subset I' of I^Z such that: $Z \subseteq \cup\{X_i : i \in I'\}$ and for all $i \in I'$, the set $\{j \in I' : X_i \cap X_j \neq \emptyset\}$ is finite, then since Z is a properly $\bigvee$-definable subset of the locally finite properly $\bigvee$-definable manifold $\mathbf{X}' = (X', (X_i', \phi_i')_{i \in I'})$ where $X' = \cup\{X_i' : i \in I'\}$ and for each $i \in I'$, $X_i' = X_i$ and $\phi_i' = \phi_i$, Z will have a properly $\bigvee$-definable cell decomposition relative to $\mathbf{X}'$. Under these conditions, we will say that Z is a *properly $\bigvee$-definable subset of X with properly $\bigvee$-definable cell decomposition*. This fact, will allow us to talk above notions in properly $\bigvee$-definable manifolds that involve the properly $\bigvee$-definable cell decomposition.

When $\mathcal{N}$ expands a real closed field, then we can use the definable triangulation theorem instead of the cell decomposition theorem. Below we include the definition of properly $\bigvee$-definable triangulation of X compatible with finitely many properly $\bigvee$-definable subsets. Similarly, all the notions that we define using cell decomposition have an analogue obtained by using the definable triangulation theorem, and moreover the two versions are compatible.

Definition 2.11 Let $S_1, \ldots, S_k \subseteq S \subseteq N^m$ be definable sets. A *definable triangulation* in N^m of S compatible with $S_1, \ldots, S_k$ is a pair (Φ, K) consisting of a complex K in N^m and a definable homeomorphism $\Phi : S \longrightarrow |K|$ such that each S_i is a union of elements of $\Phi^{-1}(K)$. We say that (Φ, K) is a *stratified definable triangulation* of S compatible with $S_1, \ldots, S_k$ if: $m = 0$ or $m > 0$ and there is a stratified definable triangulation (Ψ, L) of $\pi(S)$ compatible with $\pi(S_1), \ldots, \pi(S_k)$ (where $\pi : N^m \longrightarrow N^{m-1}$ is the projection onto the first $m-1$ coordinates) such that $\pi_{|Vert(K)} : K \longrightarrow L$ is a simplicial map and the diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & |K| \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \pi(S) & \xrightarrow{\Psi} & |L| \end{array}$$

commutes. We say that (Φ, K) is a *quasi-stratified definable triangulation* of S compatible with $S_1, \ldots, S_k$ if there is a linear bijection $\alpha : N^m \longrightarrow N^m$ such

that $(\alpha\Phi\alpha^{-1}, \alpha K)$ is a stratified definable triangulation of $\alpha(S)$ compatible with $\alpha(S_1), \ldots, \alpha(S_k)$.

Fact 2.12 *[vdd] Let $S_1, \ldots, S_k \subseteq S \subseteq N^m$ be definable sets. Then, there is a definable triangulation of S compatible with $S_1, \ldots, S_k$. Moreover, if S is bounded then, there is a quasi-stratified definable triangulation of S compatible with $S_1, \ldots, S_k$.*

Definition 2.13 Suppose that $\mathcal{N}$ expands a real closed field and let $A_1, \ldots,$ A_n, B, Z be properly $\bigvee$-definable subsets of X. Let $I = \{1, 2, \ldots\}$ be an enumeration of I. Define inductively $(X_i', (\Psi_i, M_i), (\Phi_i, N_i))$ for $i \in I$ by: $X_1' = X_1$, (Ψ_1, M_1) is a definable triangulation of $\phi_1(X_1)$ compatible with the definable subsets $\phi_1(X_1 \cap X_j)$, $\phi_1(X_1 \cap A_l)$ and $\phi_1(X_1 \cap X_j \cap A_l)$ for all $l \in \{1, \ldots, n\}$ and all $j \in I_1$, and $(\Phi_1, N_1) = (\Psi_1, M_1)$; let $X_{i+1}' := X_{i+1} \setminus \cup\{C : \exists r \in \{1, \ldots, i\}, \phi_r(C) \in N_i\}$ and be (Ψ_{i+1}, M_{i+1}) be a definable triangulation of $\phi_{i+1}(X_{i+1}')$ compatible with the definable sets $\phi_{i+1}(X_{i+1}' \cap X_j)$, $\phi_{i+1}(X_{i+1}' \cap A_l)$ and $\phi_{i+1}(X_{i+1}' \cap X_j \cap A_l)$ for all $l \in \{1, \ldots, n\}$ and all $j \in I_{i+1}$ and such that (Φ_{i+1}, N_{i+1}) which is equal to (Ψ_{i+1}, M_{i+1}) together with all the $(\Phi_{j|}, N_j)$ for $j \in I_{i+1} \cap \{1, \ldots, i\}$ is a definable triangulation of $\phi_{i+1}(X_{i+1})$.

By a properly $\bigvee$-definable triangulation of X compatible with $A_1, \ldots, A_n$ we mean a sequence $(\Phi, K) = \{(\Phi_i, N_i) : i \in I\}$ some $\{(\Phi_i, N_i) : i \in I\}$ like above. By a properly $\bigvee$-definable triangulation of B compatible with $A_1, \ldots, A_n$ we mean a sequence $(\Phi_B, K_B) = \{(\Lambda, L) \in (\Phi, K) : |L| \subseteq B\}$ for some properly $\bigvee$-definable triangulation (Φ, K) of X compatible with $A_1, \ldots, A_n, B$. Note that for each $C \in K$ such that $\phi_i(C) \in K_i$ for some $i \in I$, C is definably homeomorphic to a k-simplex. If $C \subseteq Z$ we say that C is a k-simplex of $((\Phi, K))$ in Z.

2.3 Properly $\bigvee$-definable completeness

In this subsection $\mathbf{X}$ will be a properly $\bigvee$-definable manifold. Recall that, by [ps] a Hausdorff definable manifold $\mathbf{X}$ is called *definably compact* if for every definable continuous map $\sigma : (a, b) \subseteq N \cup \{-\infty, +\infty\} \longrightarrow X$ the limits $\lim_{t \to a^+} \sigma(t)$ and $\lim_{t \to b^-} \sigma(t)$ exist in X. Here we will introduce the notion of *properly $\bigvee$-definable completeness* for properly $\bigvee$-definable subsets

of X, which coincides with the notion of definable compactness on Hausdorff definable manifolds. The notion of properly $\bigvee$-definable completeness is invariant under taking elementary extensions, elementary substructures of $\mathcal{N}$ (containing the parameters over which $\mathbf{X}$ is defined) and under taking expansions of $\mathcal{N}$ and reducts of $\mathcal{N}$ on which $\mathbf{X}$ is defined. In particular this shows that the same holds for the notion of definable compactness of definable manifolds, solving in this way a question raised (and solved for the special case of affine definable manifolds) in [ps].

For a properly $\bigvee$-definable subset Z of X, $\overline{Z}$ denotes the (topological) closure of Z in X and $\overline{Z} \setminus Z$ its boundary in X. Both $\overline{Z}$ and $\overline{Z} \setminus Z$ are properly $\bigvee$-definable subsets of X. Note that, if $C \subseteq X_i$ is a cell in X then $\overline{C}$ is a disjoint union of cells in X and $\overline{\phi_i(C)} \subseteq N^m$ is a disjoint union of cells in N^m. Below it we be convenient to work with cell in $[-\infty, +\infty]^m$. These cells are defined in the same way as the cells in N^m with the convention that the new definable sets (and maps) are the sets (and maps) definable in the structure obtained from $\mathcal{N}$ by adding the constant symbols $-\infty$ and $+\infty$. Its easy to see that the cell decomposition theorem also holds for this new structure and a cell in N^m is also a cell in $[-\infty, +\infty]$.

Definition 2.14 Let Z be a properly $\bigvee$-definable subset of X. We say that a cell C in Z is *definably complete (in Z)* if for some (equivalently for all) $i \in I$ such that $C \subseteq X_i$, for every cell of $\overline{\phi_i(C)}$ in $[-\infty, +\infty]^m$ there is a unique cell in Z contained in $\overline{C}$ of the same dimension and such that the incidence relations among the cells of $\overline{\phi_i(C)}$ in $[-\infty, +\infty]^m$ are preserved under this correspondence. We say that Z is *properly $\bigvee$-definably complete* if for each $l \in I^Z$, there is a subset I^l of I^Z such that $\mathbf{X}^l = (X^l, (X_i^l, \phi_i^l)_{i \in I^l})$ where $X^l = \cup\{X_i^l : i \in I^l\}$ and for each $i \in I^l$, $X_i^l = X_i$ and $\phi_i^l = \phi_i$ is a locally finite properly $\bigvee$-definable manifold and $Z^l := \{Z_i : i \in I^l\}$ is a properly $\bigvee$-definably subset of X^l with a properly $\bigvee$-definable cell decomposition K_l in X^l such that every cell C of K_l in Z_l is definably complete in Z^l.

Theorem 2.15 *If $\mathbf{X}$ is a Hausdorff properly $\bigvee$-definable manifold and Z is a properly $\bigvee$-definable subset of X, then Z is properly $\bigvee$-definably complete iff for every definable continuous map $\sigma : (a, b) \subseteq N \cup \{-\infty, +\infty\} \longrightarrow Z$ the limits $\lim_{t \longrightarrow a^+} \sigma(t)$ and $\lim_{t \longrightarrow b^-} \sigma(t)$ exist in Z. In particular, the notion of properly $\bigvee$-definably complete does not depend on the properly $\bigvee$-definable*

11

cell decomposition K_l of Z_l ($l \in I^Z$) and is invariant under taking elementary extensions, elementary substructures of $\mathcal{N}$ (containing the parameters over which Z is defined) and under taking expansions of $\mathcal{N}$ and reducts of $\mathcal{N}$ on which Z is defined.

Proof. By o-minimality, its enough to show that a cell $C \subseteq X_i$ in X is definably complete in X iff for every definable continuous map $\sigma : (a,b) \subseteq N \cup \{-\infty, +\infty\} \longrightarrow C$ the limit $\lim_{t \longrightarrow b^-} \sigma(t)$ exist in $\overline{C}$.

By o-minimality, $\lim_{t \longrightarrow b^-} \phi_i \circ \sigma(t)$ exists in $\overline{\phi_i(C)} \subseteq [-\infty, +\infty]^m$ (this notation means the closure in $[-\infty, +\infty]^m$) and so there is a cell $B \subseteq \overline{\phi_i(C)} \subseteq [-\infty, +\infty]^m$ containing this limit. If $B \subseteq \phi_i(X_i)$ then we are done, otherwise there is a cell $D \subseteq \overline{C}$ in K which corresponds to B and there is $j \in I$ such that $D \subseteq X_j$, $C \subseteq X_i \cap X_j$ and $\lim_{t \longrightarrow b^-} \phi_j \circ \sigma(t) \in \phi_j(D)$.

Suppose that $dim \phi_i(C) = k$. Then there is a open k-cell $B \subseteq N^k$ definably homeomorphic to $\phi_i(C)$ and by considering the definable sets $B \cap \pi_\Sigma^{-1}(x)$ for $\Sigma = (i_1, \ldots, i_{k-1})$, $\pi_\Sigma : N^k \longrightarrow N^{k-1}$ the projection onto the Σ-coordinates and $x \in \pi_\Sigma(B)$, we get a family of continuous maps $\sigma_{\Sigma,x,r} : (a_{\Sigma,x,r}, b_{\Sigma,x,r}) \longrightarrow C$ such that $\cup\{\sigma_{\Sigma,x,r}(a_{\Sigma,x,r}, b_{\Sigma,x,r}) : \Sigma, \ x \in \pi_\Sigma(B), \ r = 1, \ldots, R_{\Sigma,x}\} = C$. Using the fact that all these definable continuous maps have limits in X and the fact that the definable continuous maps $\phi_i \circ \sigma_{\Sigma,x,r}$ have limits in $\overline{\phi_i(C)} \subseteq [-\infty, +\infty]^m$ the result follows. $\square$

Note that, a properly $\bigvee$-definable subset Z of X is properly $\bigvee$-definably complete iff it is a (countable) union of definably complete (i.e., definably compact) definable subsets of X. We can use this to extend this notion to $\bigvee$-definable subsets: we say that a $\bigvee$-definable subset Z of X is $\bigvee$-*definably complete* iff it is a (countable) union of definably complete definable subsets of X. However, the $\bigvee$-definable analogue of theorem 2.15 only holds if we assume $\aleph_1$-saturation (consider again the case $\mathcal{N} = (\mathbb{Q}, <)$).

Definition 2.16 A (properly) $\bigvee$-definable subset Z of X is said to be *(properly) $\bigvee$-definably compact* if Z is (properly) $\bigvee$-definably complete and **X** is Hausdorff.

We will from now on, assume that every properly $\bigvee$-definable manifold is Hausdorff.

2.4   Properly $\bigvee$-definable connectedness

Here, **X** will be a properly $\bigvee$-definable manifold and Z a subset of X.

Definition 2.17 If Z is a properly $\bigvee$-definable subset of X. We say that Z is *properly $\bigvee$-definably connected* if there do not exist two disjoint (relatively) open properly $\bigvee$-definable subsets of Z whose union is Z. Or equivalently, if there is no open and closed proper properly $\bigvee$-definable subset of Z.

There is no interesting notion of $\bigvee$-definable connectedness for general $\bigvee$-definable subsets of X: in $\mathcal{N} = (\mathbb{Q}, <)$, $\mathbb{Q}$ is as a definable set definably connected, but is not a "$\bigvee$-definably connected" set since it is a disjoint union of the open $\bigvee$-definable subsets $(-\infty, \sqrt{2})$ and $(\sqrt{2}, +\infty)$; in an $\aleph_1$-saturated extension $\mathcal{N} = (N, <)$ of $(\mathbb{R}, <)$, the convex hull R of $\mathbb{R}$ is a proper, open and closed $\bigvee$-definable subset of the definable (and therefore also $\bigvee$-definable) set N. Note also, that the complement of R in N is not $\bigvee$-definable.

Lemma 2.18 *Let Z be properly $\bigvee$-definable subset of X. Then Z is a countable union of properly $\bigvee$-definably connected properly $\bigvee$-definable subsets.*

Proof. For each $i \in I^Z$, we have $Z_i = \cup \{Z_{i,j} : j = 1, \ldots, m_i\}$ where for each $j \in \{1, \ldots, m_i\}$, $Z_{i,j}$ is a (closed and open) definably connected component of Z_i. Let $S := \{(i, j) : i \in I^Z, \ j = 1, \ldots, m_i\}$. Clearly (by Zorn's lemma) there are disjoint subsets S_k of S with $k \in K$ such that $S = \cup \{S_k : k \in K\}$ and for each $k \in K$, $Z^k := \cup \{Z_s : s \in S_k\}$ is a properly $\bigvee$-definably connected component of Z. $\qquad\square$

By o-minimility, if $\sigma : (a, b) \subseteq N \longrightarrow X$ (with $a < b$) if a definable continuous injective map, then a total definable ordering $\leq_\sigma$ in $\sigma(a, b)$ is necessarily the ordering induced by either the ordering of (a, b) or the reverse ordering of (a, b). If $\Sigma = (\sigma(a, b), \leq_\sigma)$ is as above and $\leq_\sigma$ is induced by the ordering of (resp., the reverse ordering) of (a, b) , we define the initial point $\inf \Sigma$ of Σ to be the point $\sigma(a) := \lim_{t \longrightarrow a^+} \sigma(t)$ (resp., the point $\sigma(b) := \lim_{t \longrightarrow b^-} \sigma(t)$) and the final point of Σ denoted $\sup \Sigma$ to be the point $\sigma(b)$ (resp., $\sigma(a)$).

Definition 2.19 By a *basic definable path in $Z \subseteq X$* we mean either the *constant path ϵ_x at $x \in Z$* or a definable totally ordered set $\Sigma = (\Sigma, \leq_\Sigma)$

contained in Z such that $\Sigma = \sigma((a,b))$ for some continuous definable injective map $\sigma : (a,b) \subseteq N \longrightarrow X$ (with $-\infty < a < b < +\infty$) and $\inf \Sigma, \sup \Sigma \in Z$. We call $\sigma : (a,b) \longrightarrow X$ a *definable parametrisation of* Σ; the *support of* Σ denoted by $|\Sigma|$ is the closed set $\{\inf \Sigma\} \cup \Sigma \cup \{\sup \Sigma\}$; the *inverse* of the basic definable path Σ is the basic definable path $\Sigma^{-1} := (\Sigma, \leq_{\Sigma^{-1}})$ where for all $s, t \in \Sigma$, $s \leq_{\Sigma^{-1}} t$ iff $t \leq_{\Sigma} s$.

For convenience, if Σ is the basic definable path ϵ_x, we call $\sigma : \emptyset \longrightarrow X$ the definable parametrisation of Σ; x the support of Σ (denoted as well $|\Sigma|$); x the initial and final point of Σ (denoted as before $\inf \Sigma$ and $\sup \Sigma$); and $\Sigma^{-1} = \Sigma$.

Definition 2.20 A *definable path in $Z \subseteq X$ from x to y* is a sequence $\Sigma = \Sigma_1 \cdot \cdots \cdot \Sigma_l$ of basic definable paths Σ_i's in $Z \subseteq X$ such that $\inf \Sigma_1 = x$, $\sup \Sigma_l = y$ and for each $i = 1, \ldots, l-1$ we have $\sup \Sigma_i = \inf \Sigma_{i+1}$. x and y are also called the initial and the final point of Σ respectively (we use the notation $x = \inf \Sigma$ and $y = \sup \Sigma$); if $\inf \Sigma = \sup \Sigma = z$ we say that Σ is a *definable loop in $Z \subseteq X$ at z*; the sequence $\{\sigma_i : i = 1, \ldots, l\}$ where σ_i is a definable parametrisation of Σ_i is called a definable parametrisation (of length l) of Σ; the *support* of Σ denoted $|\Sigma|$ is $\cup\{|\Sigma_i| : i = 1, \ldots, l\}$; the *inverse* of Σ is the definable path $\Sigma^{-1} := \Sigma_l^{-1} \cdot \cdots \cdot \Sigma_1^{-1}$ in $Z \subseteq X$. We also have a natural definition of the *product* $\Sigma \cdot \Gamma$ of two definable paths Σ and Γ such that $\sup \Sigma = \inf \Gamma$.

Finally, given two definable paths $\Sigma = \Sigma_1 \cdots \cdot \Sigma_l$ and $\Gamma = \Gamma_1 \cdots \cdot \Gamma_k$ where Σ_i's and Γ_j's are basic definable paths, we define $\Sigma \simeq \Gamma$ iff $|\Sigma| = |\Gamma|$ and for all $i = 1, \ldots, l$ and $j = 1, \ldots, k$ we have $(\Sigma_i \cap \Gamma_j, \leq_{\Sigma_i}) = (\Sigma_i \cap \Gamma_j, \leq_{\Gamma_j})$. This is an equivalence relation, and we will some times identify definable paths under this equivalence relation. If $f : Z \subseteq X \longrightarrow Y$ is a strictly properly $\bigvee$-definable map and Σ is a definable path in $Z \subseteq X$ from x to y, then o-minimality implies that up to $\simeq$, there a unique definable path $f \circ \Sigma$ in $f(Z) \subseteq Y$ from $f(x)$ to $f(y)$ with $|f \circ \Sigma| = f(|\Sigma|)$.

Remark 2.21 Of course if $\mathcal{N}$ expands an ordered group then, for every definable path Σ in $Z \subseteq X$ from x to y, we have $|\Sigma| = \tau([a,b])$ for some definable continuous map $\tau : [a,b] \longrightarrow X$. If in addition, $\mathcal{N}$ expands an ordered field then we can take $[a,b] = [0,1]$.

14

Definition 2.22 We say that $Z \subseteq X$ is *definably path connected* if for any two points $y, z \in Z$ there is a definable path in Z with initial point y and final point z. $Z \subseteq X$ is *locally definably path connected* if every point in Z has a definable open neighbourhood (in Z) which is definably path connected.

Note that a properly $\bigvee$-definably connected properly $\bigvee$-definable subset Z of X is not necessarily definably path connected: take $\mathcal{N} = (\mathbb{Q}, <)$, $X = \mathbb{Q}^2$ and $Z = \{(x, y) \in \mathbb{Q}^2 : 0 < y < x\} \cup \{(0, 0), (1, 1)\} \subseteq X$; However, because the support of a definable path in X is a definably connected definable subset of X, a definably path connected properly $\bigvee$-definable subset Z of X is properly $\bigvee$-definably connected.

We now introduce two definitions that will play an important role in this paper. First recall that if, $U \subseteq X$ and $Z \subseteq X$ be a definable subset of X and a properly $\bigvee$-definable subset of X respectively. A *definable family of definable subsets of U* is simply a definable subset F of $U \times Y$. Note that $W := \pi(F)$ (where $\pi : X \times Y \longrightarrow Y$ is the projection onto Y) is a definable subset of Y and for each $w \in W$ the fibre $F_w := \{u \in U : (u, w) \in F\}$ is a definable subset of U. We use the notation $(F_w)_{w \in W}$.

Definition 2.23 We say that a (properly) $\bigvee$-definable subset Z of X has *definable choice* if for every definable family $(F_w)_{w \in W}$ of definable subsets of Z there is a definable map $s : W \longrightarrow Z$ such that for every $w \in W$, $s(w) \in F_w$. If moreover, for every definable family $(F_w)_{w \in W}$ of definable subsets of Z there is a definable map $s : W \longrightarrow Z$ such that for every $w, v \in W$, $s(w) \in F_w$ and $F_w = F_v$ iff $s(w) = s(v)$ then we say that Z has *strong definable choice*. Note that, these notions are invariant under taking elementary extensions or elementary substructures of $\mathcal{N}$ which contain the parameter over which Z is defined.

The next result was proved in [e1] in the definable case. The proof there can easily be adapted to get the following

Fact 2.24 *If Z is a properly $\bigvee$-definably complete properly $\bigvee$-definable subset of X and $(F_w)_{w \in W}$ is a definable family of definable closed subsets of Z, then there is a definable map $s : W \longrightarrow Z$ such that for all $u, v \in W$, $s(u) \in F_u$ and $F_u = F_v$ iff $s(u) = s(v)$.*

15

Corollary 2.25 *If $f : X \longrightarrow Y$ is a strictly properly $\bigvee$-definable continuous map and Z is a definably compact definable subset of X, then $f(Z)$ is a definably compact definable subset of $f(X)$.*

Proof. If $\alpha : (a, b) \longrightarrow f(Z)$ is a definable continuous map, then by fact 2.24, there is a definable map $s : (a, b) \longrightarrow Z$, such that for each $x \in (a, b)$, $s(x) \in f^{-1}(\alpha(x))$. Now since Z is definably compact the limit $\lim_{x \longrightarrow a^+} s(x)$ (resp., $\lim_{x \longrightarrow b^-} s(x)$) exist in Z and, since f is continuous and $\alpha(x) = f(s(x))$, the limit $\lim_{x \longrightarrow a^+} \alpha(x)$ (resp., $\lim_{x \longrightarrow b^-} \alpha(x)$) exist in $f(Z)$. $\square$

A *properly $\bigvee$-definable family of properly $\bigvee$-definable subsets of Z* is a properly $\bigvee$-definable subset $F \subseteq Z \times Y$. Note that, for each $(i, j) \in (I \times J)^F$, $F_{(i,j)}$ is a definable subset of F and $W_{(i,j)} := \pi(F_{(i,j)})$ is a definable subset of Y. Consider the $\bigvee$-definable subset $W := \cup\{W_{(i,j)} : (i, j) \in (I \times J)^F\}$ of Y. Then for each $w \in W$ the fibre $F_w := \{u \in Z : (u, w) \in F\}$ is a properly $\bigvee$-definable subset of Z. We use the notation $(F_w)_{w \in W}$. If for each $w \in W$, F_w is a definable subset of Z, then we say that $(F_w)_{w \in W}$ is a *properly $\bigvee$-definable family of definable subsets of Z*. Finally, if Z is a $\bigvee$-definable subset of X, by a *$\bigvee$-definable family of $\bigvee$-definable (resp., properly $\bigvee$-definable and definable) subsets of Z* we mean a $\bigvee$-definable subset F of $Z \times Y$ such that if $w \in W$ where $W := \pi(F)$ then $F_w := \{u \in Z : (u, w) \in F\}$ is a $\bigvee$-definable (resp., properly $\bigvee$-definable and definable) subset of Z.

Definition 2.26 Suppose that $Z \subseteq X$ is a (properly) $\bigvee$-definable subset. A *(properly) $\bigvee$-definable system of definable paths in Z* is a (properly) $\bigvee$-definable family $\Gamma \subseteq Z \times Z \times Z$ such that for each $(u, v) \in Z \times Z$, the fibre $\Gamma_{u,v}$ of Γ at (u, v) is a definable path in Z from u to v. We will often use the notation $\{\Gamma_{u,v} : u, v \in Z\}$ for a (properly) $\bigvee$-definable system of definable paths in Z.

Lemma 2.27 *Let $U \subseteq N^k$ be an open definable subset with definable choice and let B be a definable subset of U. Then B is definably connected iff B has a definable system of definable paths iff B is definably path connected.*

Proof. So suppose that B is definably connected. We prove the result by induction on k. The case $k = 1$ is obvious. Suppose that the result

holds in N^{k-1}. We now prove it in N^k. First suppose that B is a cell, and without loss of generality we can assume that B is an open cell in $U \subseteq N^k$ (for otherwise, $dimB < k$ and B is definably homeomorphic to an open cell in $V \subseteq N^{dimB}$) with V an open definable set definably homomorphic to U. Then we have $B = (f, g)_C$ where C is the projection of B in N^k and $f, g : C \longrightarrow N \cup \{-\infty, +\infty\}$ are continuous definable functions such that $f < g$. Since C is definably connected, it has a definable system $\{\Gamma^C_{u,v} : u, v \in C\}$ of definable paths in C. By definable choice and cell decomposition, there is a cell decomposition $C_1, \ldots, C_m$ of C such that for $i = 1, \ldots, m-1$ either C_i intersects the closure of C_{i+1} or C_{i+1} intersects the closure of C_i, together with definable continuous injective maps $\rho_i : C_i \longrightarrow B$ such that $\pi \circ \rho_i = 1_{C_i}$ where $\pi : N^k \longrightarrow N^{k-1}$ is the projection onto the first $k-1$ coordinates. The cell decomposition $C_1, \ldots, C_m$ of C induces a cell decomposition $B_1, \ldots, B_m$ of B such that for $i = 1, \ldots, m-1$ either B_i intersects the closure of B_{i+1} or B_{i+1} intersects the closure of B_i. Taking the products of "vertical paths" together the definable path in the definable system $\{\rho_i \circ \Gamma^C_{\rho_i(u), \rho_i(v)} : u, v \in C_i\}$ of definable paths in $\rho_i(C_i)$, its clear that each B_i has a definable system of definable paths. On the other hand, by definable choice in U, there is a definable path in $B_i \cup B_{i+1}$ connecting a point of B_i with a point of B_{i+1} therefore, B has a definable system of definable paths.

If B is not a cell, then B is a union of cells $C_1, \ldots, C_k$ where for each $i < k$, either C_i intersects the closure of C_{i+1}, or C_{i+1} intersects the closure of C_i and so, by definable choice in U, there is a definable path in $C_i \cup C_{i+1}$ connecting a point of C_i with a point of C_{i+1}. Since the result holds for each C_i, it also holds for B.

Clearly, if B has a definable system of definable paths, B is definably path connected. The above argument and cell decomposition show that if B is definably path connected then B is definably connected. $\qquad \square$

Proposition 2.28 *Suppose that Z is a properly $\bigvee$-definable subset of X with definable choice. Then Z is properly $\bigvee$-definably connected iff Z has a properly $\bigvee$-definable system of definable paths iff Z is definably path connected.*

Proof. For each $i \in I^Z$, $Z_i = \cup\{Z_{i,1}, \ldots Z_{i,m_i}\}$ where for each $j \in \{1, \ldots, m_i\}$, $Z_{i,j}$ is a definably connected component of Z_i. Let $S := \{(i, j) : i \in I^Z, j = 1, \ldots, m_i\}$. Then by lemma 2.27, for each $s \in S$, there is a

definable system $\{\Gamma^s_{x,y} : x, y \in Z_s\}$ of definable paths in Z_s. Let $z \in Z$ and for each $s \in S$, let $z_s \in Z_s$. To finish its enough to find definable paths Γ_s's from z to z_s. Let R be the subset of S of all s's for which Γ_s exists. By the above, both $\bigcup_{s \in R} Z_s$ and $\bigcup_{s \in S \setminus R} Z_s$ are disjoint open properly $\bigvee$-definable subsets of Z whose union is Z. Therefore, Z is properly $\bigvee$-definably connected iff $R = S$ iff Z has a properly $\bigvee$-definable system of definable paths iff Z is definably path connected. $\qquad\square$

2.5 Dimension of $\bigvee$-definable subsets

The results in this subsection are obtained by easy modifications of similar results from [p1] for definable sets and definable maps.

Lemma 2.29 *Let $Z \subseteq X$ be a properly $\bigvee$-definable subset of X (over B) and define the dimension $dimZ$ of Z to be $\max\{dimZ_i : i \in I^Z\}$. Then the following holds: (1) for $k \leq m$, $dimZ \geq k$ iff there is $i \in I^Z$ such that $dimZ_i \geq k$ iff some projection of $\phi_i(Z_i)$ onto N^k has interior in N^k iff there is a definable equivalence relation on Z_i (over B) infinitely many equivalence classes of which have dimension $\geq k - 1$ iff there is a definable subset U_i of Z_i (over B) such that $dimU_i \geq k - 1$ and U_i has no interior in Z iff there is a properly $\bigvee$-definable subset U of Z such that $dimU \geq k - 1$ and U has no interior in Z iff there is a properly $\bigvee$-definable equivalence relation on Z with properly $\bigvee$-definable classes and with infinitely many classes of dimension $\geq k - 1$ on some Z_i; (2) if $f : Z \longrightarrow Y$ is a (strictly) properly $\bigvee$-definable injective map then for each $i \in I^Z$, $dimZ_i = dimf(Z_i)$; (3) if $\{Z^s : s \in S\}$ with $|S| < \aleph_0$ is a collection of properly $\bigvee$-definable subsets of X then, $dim(\cup\{Z^s : s \in S\}) = \max\{dimZ^s : s \in S\}$ and (4) if $(F_w)_{w \in W}$ is a properly $\bigvee$-definable family of (properly) $\bigvee$-definable subsets of Z (over B) then, the set $\{w \in W : dimF_w = k\}$ is a $\bigvee$-definable subset of W (over B).*

Proof. (1) The first equivalence follows from the definition; the second equivalence is lemma 1.4 in [p1]; the third equivalence, is proposition 1.8 in [p1]; the fourth equivalence is proposition 1.9 in [p1]; the fifth equivalence follows from the previous ones by taking $U := \cup\{U_i : i \in I'\}$ where I' is the subset of I^Z of all i's such that $dimZ_i \geq k$. Now suppose that there is a properly $\bigvee$-definable subset U of Z such that $dimU \geq k - 1$ and U has no

interior in Z. By the above, we can assume without loss of generality that for each $i \in I^U$, $dimU_i \geq k-1$ and U_i has no interior in Z. Consider the following properly $\bigvee$-definable equivalence relation on Z given by: $x \sim y$ iff $x, y \in \cup\{Z_i : i \in I^U\}$ and there are $\{i_1, \ldots, i_m\} \subseteq I^U$ and $\{x = x_1, x_2, \ldots x_m = y\} \subseteq \cup\{Z_i : i \in I^U\}$ such that for each $j = 1, \ldots, m-1$, we have $x_j \sim_{i_j} x_j$ where $\sim_{i_j}$ is the definable equivalence relation on Z_{i_j} given by the fourth equivalence, or otherwise. Then, $\sim$ is a properly $\bigvee$-definable equivalence relation on Z with properly $\bigvee$-definable classes such that for some $i \in I^Z$, $\sim$ has infinitely many classes of dimension $\geq k-1$ on some Z_i. The converse is immediate, by definition of $dimZ$ and previous equivalences.

Finally, (2) and (3) are lemma 1.5 in [p1], and (4) follows from lemma 1.6 in [p1]. $\qquad\square$

Recall that, given $A \subseteq N$, $a \in N$ is in the (model theoretic) *algebraic closure of* A denoted $a \in acl(A)$, if a lies in a finite set definable over A, equivalently, a is in the *definable closure of* A denoted $a \in dcl(A)$ i.e., $\{a\}$ is definable over A. We obtain in this way an operator $acl(-) : \mathcal{P}(N) \longrightarrow \mathcal{P}(N)$, where $\mathcal{P}(N)$ is the set of all subsets of N. By the monotonicity theorem and uniform bounds, we get (see [PiS1]) that $\mathcal{N}$ is a *geometric structure* i.e., for any formula $\phi(x, y)$ there is $n \in \mathbb{N}$ such that for any $b \in N^l$, either $\phi(x, b)$ has less than n solutions in N^m or it has infinitely many, and $(N, acl(-))$ is a *pregeometry* (which means that $acl(-) : \mathcal{P}(N) \longrightarrow \mathcal{P}(N)$ satisfies the following: (*i*) if $A \subseteq N$ then $A \subseteq acl(A)$, (*ii*) if $A \subseteq N$ and $a \in acl(A)$ then, there is a finite $B \subseteq A$ such that $a \in acl(B)$, (*iii*) if $A \subseteq N$ then $acl(acl(A)) = acl(A)$ and (*iv*) if $a, b \in N$, $A \subseteq N$ and $b \in acl(A, a)$ then either $b \in acl(A)$ or $a \in acl(A, b)$ (exchange principle).

Lemma 2.30 *For $B \subseteq N$ and $a \in N^m$ let $dim(a/B) := \min\{|a'| : a' \subseteq a, \ a \subseteq acl(B \cup a')\}$. If $Z \subseteq X$ is a properly $\bigvee$-definable subset of X (over B) then $dim(Z) = \max\{dim(\phi_i(a)/B) : a \in Z_i, \ i \in I^Z\}$.*

Proof. This follows from the definition of $dimZ$ and the corresponding result for definable sets, see lemma 1.4 in [p1]. $\qquad\square$

This approach gives a rise to a good notion of dimension for general $\bigvee$-definable subsets of X when $\mathcal{N}$ is $\aleph_1$-saturated and which is coherent with the definition of dimension for properly $\bigvee$-definable subsets (for details see [p1] where the definable case is treated):

Lemma 2.31 *Suppose that $\mathcal{N}$ is $\aleph_1$-saturated and let $Z \subseteq X$ be a $\bigvee$-definable subset of X (over B). Let $dim Z := \max\{dim(\phi_i(a)/B) : a \in Z_i,\ i \in I^Z\}$. Then the following holds: (1) for $k \leq m$, $dim Z \geq k$ iff there is a definable subset W of Z such that $dim W \geq k$ iff there are $i \in I^W$ such that $dim W_i \geq k$ iff some projection of $\phi_i(W_i)$ onto N^k has interior in N^k iff there is a definable equivalence relation on W_i (over B) infinitely many equivalence classes of which have dimension $\geq k - 1$ iff there is a definable subset U_i of W_i (over B) such that $dim U_i \geq k - 1$ and U_i has no interior in Z iff there is a $\bigvee$-definable subset U of Z such that $dim U \geq k - 1$ and U has no interior in Z iff there is a $\bigvee$-definable equivalence relation on Z with $\bigvee$-definable classes and with infinitely many classes of dimension $\geq k - 1$ on some definable subset W of Z; (2) if $f : Z \longrightarrow Y$ is a (strictly) $\bigvee$-definable injective map then $dim Z = dim f(Z)$; (3) if $\{Z^s : s \in S\}$ with $|S| < \aleph_1$ is a collection of $\bigvee$-definable subsets of X then, $dim(\cup\{Z^s : s \in S\}) = \max\{dim Z^s : s \in S\}$ and (4) if $(F_w)_{w \in W}$ is a $\bigvee$-definable family of $\bigvee$-definable subsets of Z (over B) then, the set $\{w \in W : dim F_w = k\}$ is a $\bigvee$-definable subset of W (over B).*

Proof. (2) and (3) are lemma 1.5 in [p1], and (4) follows from lemma 1.6 in [p1]. Using this and the fact that $Z = \cup\{Z^s : s \in S\}$ with $|S| < \aleph_1$ and for each $s \in S$, Z^s is a properly $\bigvee$-definable subset of X, (1) follows from the corresponding result for properly $\bigvee$-definable subsets of X. $\qquad\square$

Given properly $\bigvee$-definable subsets $V \subseteq Z \subseteq X$, we say that V is *large in Z* if $dim(Z \setminus V) < dim Z$. Lemma 1.6 in [p1] implies the following result:

Lemma 2.32 *Let $Z \subseteq X$ be a properly $\bigvee$-definable subset of X (over B). If $(F_w)_{w \in W}$ is a properly $\bigvee$-definable family of properly $\bigvee$-definable subsets of Z (over B) then the set $\{w \in W : F_w \text{ is large in } Z\}$ is a $\bigvee$-definable subset of W (over B).*

If $Z \subseteq X$ is a (properly) $\bigvee$-definable subset of X (over B) and $a \in Z$ then, we say that a is a *generic point of Z over B* if $dim(\phi_i(a)/B) = dim(Z)$ for some (for every) $i \in I^Z$ such that $a \in Z_i$. Note that if $\mathcal{N}$ is $\aleph_1$-saturated, then generic points of Z over B exist. The following is an easy consequence of the definitions:

Lemma 2.33 *Let $V \subseteq Z \subseteq X$ be properly $\bigvee$-definable subsets. Then V is large in Z iff for every B over which V, Z and X are defined, every generic point of Z over B is in V.*

In general, given $\bigvee$-definable subsets $V \subseteq Z \subseteq X$, we will say that V is large in Z iff for every B over which V, Z and X are defined, every generic point of Z over B is in V. It easy to see that with this definition, we also get the analogue of lemma 2.32 for $\bigvee$-definable subsets.

3   The fundamental group

Throughout this section, $\mathbf{X} = (X, (X_i, \phi_i)_{i \in I})$ and $\mathbf{Y} = (Y, (Y_j, \psi_j)_{j \in J})$ will be properly $\bigvee$- definably connected properly $\bigvee$-definable manifolds with definable choice and of dimension m and n respectively. Note that, although we work in this section in the category of properly $\bigvee$-definable manifolds with strictly (properly) $\bigvee$-definable continuous maps, all the results we present here also hold in the category of properly $\bigvee$-definable topological spaces with corresponding strictly properly $\bigvee$-definable maps, where by a properly $\bigvee$-definable topological space we mean a properly $\bigvee$-definable subset of a properly $\bigvee$-definable manifold with the induced topology. In fact, as one can easily check, these results, except those from subsection 3.3 which use the properly $\bigvee$-definable cell decomposition theorem, also hold under the assumption that $\mathcal{N}$ is $\aleph_1$-saturated, in the category of $\bigvee$-definable topological spaces with corresponding strictly $\bigvee$-definable continuous maps, where by a $\bigvee$-definable topological space we mean a $\bigvee$-definable subset of a properly $\bigvee$-definable manifold with the induced topology.

3.1   The fundamental group

Definition 3.1 A k-cell ($k = 0, 1, 2$) with an ordered numbering of the 0-cells in X is a pair (H, ζ_H) where H is a definably complete k-cell in X and ζ_H is a numbering of the 0-cell of $\overline{H}$ (the "corners of $\overline{H}$") which induces an orientation of the boundary $\overline{H} \setminus H$ of H. Note that $|\zeta_H|$ is the number of 0-cells in $\overline{H}$.

Given $n \in \mathbb{N}$, the function $[n] : \{1, \ldots, 2n\} \longrightarrow \{1, \ldots, n\}$ is defined by: $[n](m) = m$ if $m \leq n$ and $[n](m) = m - n$ otherwise.

Definition 3.2 Let (H, ζ_H) be a k-cell ($k = 0, 1, 2$) in X_i with an ordered numbering of the 0-cells. The *special basic definable paths of* (H, ζ_H) are the basic definable paths $h_{i,j}$ in $\overline{H} \setminus H$ joining the 0-cell i to the 0-cell j where $i = 1, \ldots, |\zeta_H|$ and $j = i$ or $j = [|\zeta_H|](i + 1)$. The *special definable paths of* (H, ζ_H) are the following definable paths: (1) special basic definable path of (H, ζ_H); (2) $h_{k,k} \cdot h_{k,j}$ and $h_{k,j} \cdot h_{j,j}$; (3) $h_{k,[|\zeta|](k+1)} \cdot h_{[|\zeta_H|](k+1),[|\zeta_H|](k+2)} \cdots \cdots h_{[|\zeta_H|](k+j-1),[|\zeta_H|](k+j)}$ for $k, j \in \{1, \ldots, |\zeta_H|\}$; and (4) Σ^{-1} where Σ is a special definable path of (H, ζ_H).

We say that two definable paths $\Sigma = \Sigma_1 \cdot \cdots \cdot \Sigma_l$ and $\Gamma = \Gamma_1 \cdot \cdots \cdot \Gamma_l$ in X are *adjacent* if for all $j = 1, \ldots, l$ Σ_j and Γ_j are definable paths with $|\Sigma_j|, |\Gamma_j| \subseteq \overline{X_{j_i}}$.

Definition 3.3 Given two definable paths Σ and Γ in X, we define $\Sigma \sim^0 \Gamma$ iff one of the following holds: (1) there is (H, ζ_H) a k-cell ($k = 0, 1, 2$) in X_i with an ordered numbering of the 0-cells and there are special definable paths Σ' and Γ' of (H, ζ_H) such that $\Sigma \simeq \Sigma'$, $\Gamma \simeq \Gamma'$, $\inf \Sigma' = \inf \Gamma'$ and $\sup \Sigma' = \sup \Gamma'$, in this case we call H a *basic definable pre-homotopy* and we write $\Sigma \sim_H^0 \Gamma$; (2) there are two adjacent definable paths $\Sigma' = \Sigma_1 \cdot \cdots \cdot \Sigma_l$ and $\Gamma' = \Gamma_1 \cdot \cdots \cdot \Gamma_l$ in X such that $\Sigma \simeq \Sigma'$ and $\Gamma \simeq \Gamma'$, and there is a sequence $H = \{H_j : j = 1, \ldots, l\}$ of basic definable pre-homotopies $\Sigma_j \sim_{H_j}^0 \Gamma_j$.

If (1) or (2) holds, we write $\Sigma \sim_H^0 \Gamma$ and say that H is a *definable pre-homotopy* of Σ and Γ.

Finally we are ready to the define the notion of definable homotopy between definable paths in X. Note that, by the definition of $\sim^0$, if $\Sigma \sim^0 \Gamma$ then we have $\inf \Sigma = \inf \Gamma$ and $\sup \Sigma = \sup \Gamma$.

Definition 3.4 Given two definable paths Σ and Γ in X we define $\Sigma \sim \Gamma$ iff one of the following holds: (1) there is a sequence of definable paths $\{\Sigma_j : j = 1, \ldots, l\}$ such that $\Sigma_1 \simeq \Sigma$, $\Sigma_l \simeq \Gamma$ and there is a sequence $H = \{H_j : j = 1, \ldots, l - 1\}$ of definable pre-homotopies $\Sigma_j \sim_{H_j}^0 \Sigma_{j+1}$, in this case we write $\Sigma \sim_H \Gamma$ and say that H is a *definable homotopy with fixed endpoints* of Σ and Γ; (2) there are definable paths Λ_1 and Λ_2 such that $\Sigma \sim_H \Lambda_1 \cdot \Gamma \cdot \Lambda_2$ for some definable homotopy with fixed endpoints H.

If (1) or (2) holds, we write $\Sigma \sim_H \Gamma$ and say that H is a *definable homotopy* of Σ and Γ.

Definition 3.5 Suppose that $\Sigma \sim_B \Gamma$ with $B = \{B_i : i = 1, \ldots, n\}$, $\Sigma_i \sim_{B_i} \Sigma_{i+1}$ for $i = 1, \ldots, m-1$ and $\Sigma \simeq \Sigma_1$ and $\Gamma \simeq \Sigma_m$ is a definable homotopy. The *support* $|B|$ of the definable homotopy B is the union of the closure of all k-cells ($k = 0, 1, 2$) in B. Let K be a finite collection of properly $\bigvee$-definable sets in X. A *refinement B' of B respecting K* is a definable homotopy $\Sigma \sim_{B'} \Gamma$ obtained by taking a cell decomposition of $|B|$ compatible with the definable sets: $|\Sigma_i|$, all the cells in the closure of B_i and $B_i \cap N$ for all $i \in \{1, \ldots, n\}$ and all definable sets N of K, and modifying the definable homotopy B by adding the new k-cells ($k = 0, 1, 2$), modifying the definable paths Σ_i and enlarge the list. We will often identify two definable homotopies B and B' under the equivalence relation $B \simeq B'$ iff B and B' have a common refinement.

If $\Sigma \sim_H \Gamma$ and $\inf \Sigma = \inf \Gamma$ we say that Σ and Γ are definably homotopic with fixed initial point denoted $\Sigma \sim_H \Gamma(rel\{\inf\})$. Similarly we define definably homotopic with fixed final point and definably homotopic with fixed end points and call H a definable homotopy with fixed final point, end points respectively and write $\Sigma \sim_H \Gamma(rel\{\sup\})$ and $\Sigma \sim_H \Gamma(rel\{\inf, \sup\})$. As before, we sometimes omit the subscript H in $\sim_H$. More generally and similarly, given properly $\bigvee$-definable subsets A and B of X, we can define the notion of relative definable homotopy of definable paths Σ and Γ with initial point in A and final point in B, denoted $\Sigma \sim_H \Gamma(rel\{A, B\})$. It follows easily from the definitions that all these relations are in fact equivalence relations, we denote the set of equivalence classes by $\pi_1(X, A, B)$ and the equivalence class of Σ is denoted by $[\Sigma]$. As usual, we write $\pi_1(X, A)$ for $\pi_1(X, A, A)$ and $\pi_1(X, x)$ for $\pi_1(X, \{x\})$.

Lemma 3.6 *Let Σ and Γ be two definable paths in X and let $f : X \longrightarrow Y$ be a strictly properly $\bigvee$-definable continuous map. If $\Sigma \sim \Gamma$ then $f \circ \Sigma \sim f \circ \Gamma$.*

Proof. The proof is of course by induction on the definition of $\sim$. There result will therefore follow, if we prove it in the case where there is (H, ζ_H) a k-cell ($k = 0, 1, 2$) in X_i with an ordered numbering of the 0-cells, and Σ and Γ are special definable paths of (H, ζ_H) such that $\Sigma \sim_H^0 \Gamma$.

Let J' be a finite subset of J such that $f(H) \subseteq \{Y_j : j \in J'\}$. If $k = 0, 1$ the result is obvious, so we can assume that $k = 2$. We now prove the result by induction on $|J'|$.

Suppose that $|J'| = 1$. Then using the fact that there is a definable homeomorphism $\rho : (g, h)_{(a,b)} \subseteq N^2 \longrightarrow H$ where $g, h : (a, b) \longrightarrow N$ are

definable continuous maps such that $g < h$, and using cell decomposition its easy to see that there is a cell decomposition C of $f(H)$ such that $f \circ \Sigma \sim_C f \circ \Gamma$: Let K be a cell decomposition of $f(H)$, then there is a cell decomposition K' of H such that for each cell $B \in K'$, $f(B)$ is a cell in $f(H)$. Take C to be the cell decomposition of $f(H)$ compatible with K and each cell $f(B)$ where $B \in K'$.

For the case $|J'| > 1$, we first show that there is a cell decomposition $C = \{C_1, \ldots, C_q\}$ of H such that $\Sigma \sim_C \Gamma$ and for each $i = 1, \ldots, q$ there is $j \in J'$ such that $C_i \subseteq Y_j$. Let $K = \{K_1, \ldots, K_p\}$ be a cell decomposition of H compatible with the definable sets $f^{-1}(f(H) \cap Y_j)$ where $j \in J'$. Using ρ and o-minimality we can refine the cell decomposition K if necessary, by adding points $a = a_1 < a_2 < \cdots < a_t = b$ to $[a, b]$ to get a cell decomposition C such that: each 0-cell in C is a 0-cell of some $[\rho \circ g(a_i), \rho \circ h(a_i)]$; each 1-cell in C is either a subinterval of some $[\rho \circ g(a_i), \rho \circ h(a_i)]$ or the graph of a definable continuous injective function $\rho \circ k : (a_i, a_{i+1}) \longrightarrow H$; and each a 2-cell in C is of the form $\rho((k, k')_{(a_i, a_{i+1})})$ for some definable continuous injective functions $k, k' : (a_i, a_{i+1}) \longrightarrow \rho^{-1}(H)$ such that $k < k'$. Its clear now that this (refined) cell decomposition C of H satisfies the claim.

Therefore, we have reduced the case $|J'| > 1$ to the case where, $\Sigma \sim_C \Gamma$ and each definable pre-homotopy $\Sigma_i \sim^0_{C_i} \Gamma_i$ which occurs here is such that $f(C_i) \subseteq Y_j$ for some j. But then, induction and the case $|J'| = 1$ proves the result for $|J'| > 1$. $\qquad \square$

Given $A_1, \ldots, A_l \subseteq X'$ and $B_1, \ldots, B_l \subseteq Y'$ properly $\bigvee$-definable subsets of X and Y respectively and a continuous strictly properly $\bigvee$-definable map $f : X' \longrightarrow Y'$ such that $f(A_i) \subseteq B_i$, we write $f : (X', A_1, \ldots, A_l) \longrightarrow (Y', B_1, \ldots, B_l)$ (if $l = 1$, $A_1 = \{x\}$ and $B_1 = \{y\}$ we write $f : (X', x) \longrightarrow (Y', y)$).

Theorem 3.7 π_1 *is a covariant functor from the category of pointed properly $\bigvee$-definable manifolds into the category of groups. $\pi_1(X, x)$ is a group called the definable fundamental group of X at x with the product defined by $[\Sigma][\Gamma] := [\Sigma \cdot \Gamma]$ and given strictly properly $\bigvee$-definable map $f : (X, x) \longrightarrow (Y, y)$, $\pi_1(f)$ (which is denoted f_*) is defined by $f_*([\Sigma]) = [f \circ \Sigma]$. Moreover, $\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$ and, if there is a definable path in X from x_0 to x_1, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.*

Proof. It follows easily from the definition of definable homotopy of definable paths that $\pi_1(X, x)$ is in fact a well defined group with identity $[\epsilon_x]$ and the inverse $[\Sigma]^{-1}$ of $[\Sigma]$ given by $[\Sigma^{-1}]$. The fact that $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$ is well defined follows from lemma 3.6, and f_* is a homomorphism since $f \circ (\Sigma \cdot \Gamma) \simeq (f \circ \Sigma) \cdot (f \circ \Gamma)$.

Let Θ be a definable path from x_0 to x_1. Its easy to see that the map $\Lambda : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ given by $\Lambda([\Sigma]) := [\Theta \cdot \Sigma \cdot \Theta^{-1}]$ is a well defined isomorphism.

The isomorphism $\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$ is easy to verify: let $\pi^X : X \times Y \longrightarrow X$ and $\pi^Y : X \times Y \longrightarrow Y$ be the natural projections, and let $i^X : X \longrightarrow X \times Y$ and $i^Y : Y \longrightarrow X \times Y$ be given by $i^X(u) := (u, y)$ and $i^Y(v) := (x, v)$ respectively; let $\beta : \pi_1((X \times Y), (x, y)) \longrightarrow \pi_1(X, x) \times \pi_1(Y, y)$ be given by $\beta([\Sigma]) := (\pi^X_*([\Sigma]), \pi^Y_*([\Sigma]))$ and let $\alpha : \pi_1(X, x) \times \pi_1(Y, y) \longrightarrow \pi_1((X \times Y, (x, y))$ be given by $\alpha([\Sigma], [\Gamma]) := i^X_*([\Sigma]) \cdot i^Y_*([\Gamma])$. Then clearly, $\beta \circ \alpha = 1_{\pi_1(X,x) \times \pi_1(Y,y)}$ and therefore, β is surjective. On the other hand, if $\beta([\Sigma]) = \beta([\Gamma])$ and $\pi^X \circ \Sigma \sim_H \pi^X \circ \Gamma$ and $\pi^Y \circ \Sigma \sim_K \pi^Y \circ \Gamma$, then by considering the definable set $(\pi^X)^{-1}(|H|) \cap (\pi^Y)^{-1}(|K|)$ and cell decomposition, its easy to construct a definable homotopy $\Sigma \sim \Gamma$, and therefore, β is injective. $\square$

Definition 3.8 The properly $\bigvee$-definably connected, properly $\bigvee$-definable set X is called *definably simply connected* if $\pi_1(X, x) = 0$ for some (equivalently for all) $x \in X$.

3.2 Homotopy type

Definition 3.9 Let $f, g : (X', A_1, \ldots, A_l) \longrightarrow (Y', B_1, \ldots, B_l)$ be two continuous strictly properly $\bigvee$-definable maps. We say that f and g are *strictly properly $\bigvee$-definably pre-homotopic* if there is a continuous strictly properly $\bigvee$-definable map $H : (X' \times |\Gamma|, A_1 \times |\Gamma|, \ldots, A_l \times |\Gamma|) \longrightarrow (Y', B_1, \ldots, B_l)$ such that $H(x, \inf \Gamma) = f(x)$ and $H(x, \sup \Gamma) = g(x)$ for all $x \in X'$, where Γ is a definable path in some $\bigvee$-definable manifold $\mathbf{Z}$. H is called a *strictly properly $\bigvee$-definable pre-homotopy* between f and g. We say that f and g are *strictly properly $\bigvee$-definably homotopic* if there is a sequence $f = h_0, h_1, \ldots, h_m = g$ of continuous strictly properly $\bigvee$-definable maps from $(X', A_1, \ldots, A_l)$ into $(Y', B_1, \ldots, B_l)$ and a sequence $H = \{H_i : i = 1, \ldots, m\}$ of strictly properly

$\bigvee$-definable pre-homotopies H_i between h_{i-1} and h_i. H is called a *strictly properly $\bigvee$-definable homotopy* between f and g.

Strictly properly $\bigvee$-definable homotopy between continuous strictly properly $\bigvee$-definable maps is an equivalence relation compatible with composition in the set of all such continuous strictly properly $\bigvee$-definable maps. We denote by $[f]$ the equivalence class of $f : (X', A_1, \ldots, A_l) \longrightarrow (Y', B_1, \ldots, B_l)$ and by $[(X', A_1, \ldots, A_l), (Y', B_1, \ldots, B_l)]$ the set of all such classes. We say that $(X', A_1, \ldots, A_l)$ and $(Y', B_1, \ldots, B_l)$ have the same *strictly properly $\bigvee$-definable homotopy type* if there are continuous strictly properly $\bigvee$-definable maps $f : (X', A_1, \ldots, A_l) \longrightarrow (Y', B_1, \ldots, B_l)$ and $g : (Y', B_1, \ldots, B_l) \longrightarrow (X', A_1, \ldots, A_l)$ such that $[g \circ f] = [1_{X'}]$ and $[f \circ g] = [1_{Y'}]$. This is an equivalence relation and we denote by $[(X', A_1, \ldots, A_l)]$ the equivalence class of $(X', A_1, \ldots, A_l)$.

Fact 3.10 *Let $f, g : (X, x) \longrightarrow (Y, y)$ and $f_i : (X, x) \longrightarrow (Y, \{y_0, y_1\})$ $(i = 0, 1)$ with $f_i(x) = y_i$ be continuous strictly properly $\bigvee$-definable maps. If $[f] = [g]$, then $f_* = g_*$ and if H is a strictly properly $\bigvee$-definable homotopy between f_0 and f_1 then, there is a definable path Γ in Y from y_0 to y_1 which determines a homomorphism $\Gamma_\# : \pi_1(Y, y_0) \longrightarrow \pi_1(Y, y_1)$ such that $\Gamma_\# \circ f_{0*} = f_{1*}$. In particular, if $f : (X, x) \longrightarrow (Y, y)$ is a strictly properly $\bigvee$-definable homotopy equivalence then, the induced homomorphism $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$ is an isomorphism.*

Of course if $\mathcal{N}$ expands a real closed field, then we can take in definition 3.9 $|\Gamma|$ to be $[0, 1]$. Moreover we also have the following result:

Proposition 3.11 *Suppose that $\mathcal{N}$ is an expansion of a real closed field. Then $\pi_1(X, A, B) = [([0, 1], \{0\}, \{1\}), (X, A, B)]$.*

Proof. Let Σ and Γ be two definable paths in X. We need to show that, $\Sigma \sim \Gamma$ iff there is a definable continuous function $K : ([0, 1] \times [0, 1], \{0\} \times [0, 1], \{1\} \times [0, 1]) \longrightarrow (X, A, B)$ such that $\Sigma \simeq K_0$ and $\Gamma \simeq K_1$ where for $i = 0, 1$ K_i is the definable path in X parametrised by $t \longrightarrow K(i, t)$. The implication from right to left follows from lemma 3.6. By the transitivity of the two notions of definable homotopy and by the inductive definition of $\sim$, to show the implication from left to right, its enough to assume that there is (H, ζ_H) a k-cell $(k = 0, 1, 2)$ in X_i with an ordered numbering of the 0-cells,

and Σ and Γ are special definable paths of (H, ζ_H) such that $\Sigma \sim_H^0 \Gamma$. The cases $k = 0, 1$ are easy to prove, for the case $k = 2$ use first the fact that the closure of a 2-cell $(g, h)_{(a,b)} \subseteq N^2$ is definable homeomorphic to $[0, 1] \times [0, 1]$. $\square$

3.3 The o-minimal Tietze theorem

In this subsection, we generalise results proved in [bo] for definable sets in an o-minimal expansion of a real closed field. *We assume here that X is properly $\bigvee$-definably complete with a properly $\bigvee$-definable cell decomposition.*

Definition 3.12 Let K be a properly $\bigvee$-definable cell decomposition of X and v a 0-cell of K. An edge path of K is a sequence $\sigma = u_1, u_2, \ldots, u_l$ of 0-cells of K such that for all $i = 1, \ldots, l - 1$, u_i, u_{i+1} are 0-cells of a 1-cell of K. If $\gamma = w_1, w_2, \ldots, w_k$ is another edge path of K and u_l, w_1 are 0-cells of a 1-cell of K then the concatenation $\sigma \cdot \gamma$ is also an edge path of K. σ is an edge loop of K at v if $v = u_1 = u_l$. Note that an edge path σ of K determines uniquely a definable path Σ in X and the concatenation of two edge paths of K corresponds to the product of the corresponding definable paths. $E(K, v)$ is the group under the operation $[\sigma][\gamma] := [\sigma \cdot \gamma]$ of classes $[\sigma]$ of edge loops σ of K at v under the equivalence relation: $\sigma \sim \gamma$ iff $\Sigma \sim_H \Gamma$ where every k-cell $(k = 0, 1, 2)$ in H is a k-cell of K.

Let K be as above. A tree in K is a collection of 1-cells and 0-cells which is a tree; a maximal tree in K necessarily contains all the 0-cells of K. A special (basic) definable path of K is a special (basic) definable path of some k-cell $(k = 0, 1, 2)$ H of K.

Lemma 3.13 *Let K be a properly $\bigvee$-definable cell decomposition of X, v a 0-cell of K and T a maximal tree in K. Then $E(K, v)$ is isomorphic to the group $G(K, T)$ generated by the special basic definable paths of K with relations: (1) $\Sigma = 1$ if Σ is a special basic definable paths of K contained in T; and (2) $\Sigma = \Gamma$ if Σ and Γ are special definable paths of K and $\Sigma \sim_H^0 \Gamma$ for some k-cell $(k = 0, 1, 2)$ H of K.*

Proof. The isomorphism $\overline{\kappa} : G(K, T) \longrightarrow E(K, v)$ is induced by the map κ defined as follows: Let $h_{a,b}$ be a special basic definable path of K where

a, b are 0-cells of some 1-cell of K, then since T is definable path connected, there is a unique edge path e in T from v to a with no repetition of vertices, with inverse e^{-1} the unique edge path in T from a to v with no repetition of vertices. Define $\kappa(h_{a,b}) := e, a, b, a, e^{-1}$. Let σ be an edge loop of K at v then there is an edge loop $\gamma = u_1, u_2, \ldots, u_k$ of K at v with minimal k such that $[\sigma] = [\gamma]$. Then $[\kappa(h_{u_1,u_2} \cdot h_{u_2,u_3} \cdots h_{u_{k-1},u_k})] = [\sigma]$. Similarly, its easy to see that $\overline{\kappa}$ is injective. $\qquad\square$

Remark 3.14 . Let K be a properly $\bigvee$-definable cell decomposition of X and let C be a l-cell of K in X. Then by an easy induction argument on l one can prove that: for any definable path Σ in $\overline{C}$ there is a definable homotopy $\Sigma \sim_H \Gamma$ where Γ is a definable path contained in $\overline{C}$ obtained from edge paths of K contained in $\overline{C}$; moreover, for any definable paths Σ and Γ obtained from edge paths of K contained in $\overline{C}$ (with the same endpoints) there is a definable homotopy $\Sigma \sim_H \Gamma$ (with fixed endpoints) such that every k-cell ($k = 0, 1, 2$) of H is a k-cell of K.

Theorem 3.15 *(Tietze theorem). Let K be a properly $\bigvee$-definable cell decomposition of X and T a maximal tree in K. Then, $\pi_1(X, x) \simeq G(K, T)$. In particular, $\pi_1(X, x)$ is invariant under taking elementary extensions, elementary substructures of $\mathcal{N}$ (containing the parameters over which $\mathbf{X}$ is defined) and under taking expansions of $\mathcal{N}$ and reducts of $\mathcal{N}$ on which $\mathbf{X}$ is defined and X has definable choice.*

    Proof. By theorem 3.7 and lemma 3.13 its enough to show that: (i) every definable loop in X at v (a fixed 0-cell of K) is definably homotopic to a definable loop in X at v obtained from an edge loop of K at v and (ii) if Σ and Γ are two definable loops in X at v obtained from edge loops σ and γ of K at v such that Σ and Γ are definably homotopic then, $\Sigma \sim_H \Gamma$ where every k-cell ($k = 0, 1, 2$) in H is a k-cell of K.

    Let Σ be a definable loop in X at v. Then clearly, there are definable paths $\Sigma_1, \ldots, \Sigma_k$ and there are cells $C_1, C_2, \ldots, C_m$ of K such that: (1) $\Sigma \simeq \Sigma_1 \cdot \Sigma_2 \cdots \Sigma_k$; (2) for each $i = 1, 2, \ldots, k$ there is $l(i) \in \{1, \ldots, m\}$ such that $|\Sigma_i| \subseteq C_{l(i)}$ and (3) for each $l \in \{1, \ldots, m-1\}$, C_l is in the closure of C_{l+1} or C_{l+1} is in the closure of C_l. Its clear that, we can enlarge if necessary the

list $\Sigma_1, \ldots, \Sigma_k$ and the list $C_1, C_2, \ldots, C_m$ preserving properties (2) and (3) above, so that we have: (4) for every $i \in \{1, \ldots, m\}$ every cell of K in $\overline{C_i}$ is in $\{C_1, \ldots, C_m\}$; (5) $\Sigma \sim \Gamma_1 \cdot \Gamma_2 \cdots \Gamma_n$ where for each $j = 1, \ldots, n$ the definable path Γ_j is such that the following holds: $(5a)$ $\Gamma_j = \Sigma_{j_1} \cdot \Sigma_{j_2} \cdots \Sigma_{j_{n(j)}}$ where for each $i \in \{j_1, \ldots, j_{n(j)}\}$ there is $l(i) \in \{1, \ldots, m\}$ such that $|\Sigma_i| \subseteq C_{l(i)}$, there is $r(j) \in \{j_1, \ldots, j_{n(j)}\}$ such that the collection $C_{l(j_1)}, \ldots, C_{l(j_n)}$ is contained in the collection of all the cells of K contained in $\overline{C_{r(j)}}$, and $C_{l(j_1)}$ and $C_{l(j_n)}$ are 0-cells. To finish the proof of this part of the theorem, its enough to show that each definable path Γ_j is definably homotopic to a special basic definable path of a k-cell $(k = 0, 1)$ in the list $C_1, \ldots, C_m$. But this last claim is remark 3.14.

Let Σ and Γ be definable loops in X at v obtained from edge loops σ and γ of K at v, and suppose that $\Sigma \sim_B \Gamma$ with $B = \{B_i : i = 1, \ldots, n\}$, $\Sigma_i \sim_{B_i} \Sigma_{i+1}$ for $i = 1, \ldots, m-1$ and $\Sigma \simeq \Sigma_1$ and $\Gamma \simeq \Sigma_m$. By taking if necessary a refinement of B respecting K, we can assume that, for each $i \in \{1, \ldots, n\}$ there are cells $C_{1^i}, \ldots, C_{m^i}$ of K satisfying condition (4), each definable path Σ_i satisfies condition (5) (with Σ substituted by Σ_i, each Γ_j substituted by Γ_{j^i}, n by n^i and j by j^i) and (6) for each $u \in \{1^i, \ldots, m^i\}$ there is $v(u) \in \{1^{i+1}, \ldots, m^{i+1}\}$ such that $\overline{C_u} \cap \overline{C_{v(u)}} \neq \emptyset$. Now the theorem follows from remark 3.14. $\qquad\square$

Theorem 3.16 *(van Kampen theorem).* *Let X_0, X_1, X_2 be closed properly $\bigvee$-definably connected, properly $\bigvee$-definable subsets of X with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Let $x \in X_0$. Then for any group G and any homomorphisms $h_\alpha : \pi_1(X_\alpha, x) \longrightarrow G$ $(\alpha = 0, 1, 2)$ such that $h_0 = h_\alpha \circ i_\alpha$ where $i_\alpha : \pi_1(X_0, x) \longrightarrow \pi_1(X_\alpha, x)$ is the homomorphism induced by the inclusion, there is a unique homomorphism $h : \pi_1(X, x) \longrightarrow G$ such that $h_\alpha = h \circ j_\alpha$, where $j_\alpha : \pi_1(X_\alpha, x) \longrightarrow \pi_1(X, x)$ is the homomorphism induced by the inclusion.*

Proof. Take a properly $\bigvee$-definable cell decomposition K of X compatible with X_0, X_1 and X_2. K determines properly $\bigvee$-definable cells decompositions K_0, K_1 and K_2 respectively. Take a maximal tree T_0 in K_0 and extend it to maximal trees T_1 and T_2 in K_1 and K_2 respectively. Then, $T := T_1 \cup T_2$ is a maximal tree in K. By theorem 3.15 we have $\pi_1(X, x) = G(K, T)$ and $\pi_1(X_\alpha, x) = G(K_\alpha, T_\alpha)$. The results follows since $i_\alpha(\pi_1(X_0, x))$ is generated by the special basic definable paths of $X_0 \setminus T_0$ in X_α. $\qquad\square$

When $\mathcal{N}$ expands a real closed field, then we can use the definable triangulation theorem instead of the cell decomposition theorem.

Definition 3.17 Suppose that $\mathcal{N}$ is an o-minimal expansion of a real closed field and let (Φ, K) be a properly $\bigvee$-definable triangulation of X, v a vertex of K and T a maximal tree in K (T contains all the vertices of K). An edge path of K is a sequence $\sigma = u_1, u_2, \ldots, u_l$ of vertices of K such that for all $i = 1, \ldots, l-1$, u_i, u_{i+1} are vertices of an edge of K. If $\gamma = w_1, w_2, \ldots, w_k$ is another edge path of K and u_l, w_1 are vertices of an edge of K then the concatenation $\sigma \cdot \gamma$ is also an edge path of K. σ is an edge loop of K at v if $v = u_1 = u_l$. $E(K, v)$ is the group under the operation $[\sigma][\gamma] := [\sigma \cdot \gamma]$ of classes $[\sigma]$ of edge loops σ of K at v under the equivalence relation: $v, u, a, b, c, \ldots, w, v \sim v, u, a, c, \ldots, w, v$ iff abc spans a k-simplex of K where $k = 0, 1, 2$. $E(K, v)$ is isomorphic to the group $G(K, T)$ generated by the 1-simplexes of K, denoted $g_{a,b}$ for each edge a, b and with relations: (1) $g_{a,b} = 1$ if a, b spans a simplex of L and (2) $g_{a,b}g_{b,c} = g_{a,c}$ if a, b, c spans a simplex of K.

Corollary 3.18 *(Tietze theorem). Suppose that $\mathcal{N}$ is an o-minimal expansion of a real closed field and let (Φ, K) be a properly $\bigvee$-definable triangulation of X and T a maximal tree in K. Then, $\pi_1(X, x) \simeq G(K, T)$. In particular, $\pi_1(X, x)$ is invariant under taking elementary extensions, elementary substructures of $\mathcal{N}$ (containing the parameters over which $\mathbf{X}$ is defined) and under taking expansions of $\mathcal{N}$ and reducts of $\mathcal{N}$ on which $\mathbf{X}$ is defined and X has definable choice.*

4 $\bigvee$-definable covering spaces

For the rest of this section an less otherwise stated, $\mathbf{X} = (X, (X_i, \phi_i)_{i \in I})$, $\mathbf{Y} = (Y, (Y_j, \psi_j)_{j \in J})$ and $\mathbf{Z} = (Z, (Z_k, \tau_k)_{k \in K})$ will be properly $\bigvee$-definably connected, properly $\bigvee$-definable manifolds with definable choice. As in the last section, the results of this section also hold in the category of properly $\bigvee$-definable topological spaces with strictly properly $\bigvee$-definable continuous maps.

4.1 Strictly properly $\bigvee$-definable covering spaces

Definition 4.1 (Y, p, X) is called a *strictly properly $\bigvee$-definable covering space* if the strictly properly $\bigvee$-definable map $p : Y \longrightarrow X$ is continuous, surjective and there is a cover $\{U_l : l \in L\}$ of X with $|L| < \aleph_1$ such that: (1) for each $l \in L$, U_l is an open definably connected definable subset of X; (2) for each $i \in I$, there a finite subset L_i of L such that $X_i \subseteq \cup \{U_l : l \in L_i\}$; and (3) for each $l \in L$, $p^{-1}(U_l)$ is a disjoint union of open definable subsets of Y, each of which is mapped homeomorphically by p onto U_l. We say that $p : Y \longrightarrow X$ is a *strictly properly $\bigvee$-definable covering map* and $\{U_l : l \in L\}$ is called a *p-admissible family of definable neighbourhoods*.

If (Y, p, X) is a strictly properly $\bigvee$-definable covering space then: (1) for every definable subset Z of X, $p^{-1}(Z)$ is a properly $\bigvee$-definable subset of Y, and in particular, for each $x \in X$, $p^{-1}(x)$ is a properly $\bigvee$-definable discrete subset of Y and therefore, $|p^{-1}(x)| < \aleph_1$; (2) if for each $l \in L$, $p^{-1}(U_l)$ is a disjoint union of the open definable subsets $V_{l,s}$ of Y with $s \in S_l$, each of which is mapped homeomorphically by p onto U_l then, since for each $j \in J$, the map $p_{|Y_j} : Y_j \longrightarrow X$ is definable, there is a finite subset Q_j of $\{(l, s) : l \in L, \ s \in S_l\}$ such that $Y_j \subseteq \cup \{V_q : q \in Q_j\}$; (3) let $i \in I$, then $p^{-1}(X_i)$ is an open properly $\bigvee$-definable subset of Y and considering the open definable subsets $V_{i,s}$ of Y where $s \in \cup \{S_l : l \in L^i\}$ and L^i is a finite subset of L such that $X_i \subseteq \cup \{U_l : i \in L^i\}$, we see that there are open definable subsets $Y_{i,r}$ of $p^{-1}(X_i)$ for $r \in R_i$ with $|R_i| < \aleph_1$ such that: (i) $p^{-1}(X_i) = \cup \{Y_{i,r} : r \in R_i\}$; (ii) for each $r \in R_i$, the set $\{s \in R_i : Y_{i,s} \cap Y_{i,r} \neq \emptyset\}$ is finite; and (iii) for each $r \in R_i$, $p_{|Y_{i,r}} : Y_{i,r} \longrightarrow X_i$ is a definable surjective map. The collection $\{Y_{i,r} : i \in I, \ r \in R_i\}$ of open definable subsets of Y determine in a natural way a $\bigvee$-definable manifold $\mathbf{Y}'$ which we can identify with $\mathbf{Y}$ (they are strictly properly $\bigvee$-definably isomorphic), and under this identification (which we will be considering from now on), its easy to see that $\mathbf{X}$ is a properly $\bigvee$-definable manifold iff $\mathbf{Y}$ is a properly $\bigvee$-definable manifold; (4) the properly $\bigvee$-definable covering map $p : Y \longrightarrow X$ is an open surjection: Let V be an open $\bigvee$-definable subset of Y and let $x \in p(V)$. Let U be a p-admissible definable neighbourhood of x, $y \in p^{-1}(x) \cap V$ and let W be the definable sheet over U containing y. Then $W \cap V$ is an open definable subset of V containing y, $p(W \cap V)$ is an open definable subset of U containing x and therefore, $p(V)$ is open.

Definition 4.2 We define a strictly properly $\bigvee$-definable isomorphism between strictly properly $\bigvee$-definable covering spaces (Y, p, X) and (Z, q, X) to be a strictly properly $\bigvee$-definable homeomorphism $\phi : Y \longrightarrow Z$ such that $q \circ \phi = p$. The *group of strictly properly $\bigvee$-definable covering transformations*, $Cov(Y/X)$ is the group of all strictly properly $\bigvee$-definable homeomorphisms $\phi : Y \longrightarrow Y$ such that $p \circ \phi = p$.

Pointed strictly properly $\bigvee$-definable covering space $((Y, y), p, (X, x))$ and the group $Cov((Y, y)/(X, x))$ of pointed strictly properly $\bigvee$-definable covering transformations are define in the obvious way.

Remark 4.3 Let (Y, p, X) be a strictly properly $\bigvee$-definable covering space and let W be a properly $\bigvee$-definable subset of X such that W or $p^{-1}(W)$ has a properly $\bigvee$-definable cell decomposition. Then there are properly $\bigvee$-definable cell decompositions L and K of $p^{-1}(W)$ and W respectively such that for every cell C of L, $p(C)$ is a cell of K and $p_{|C} : C \longrightarrow p(C)$ is a definable homeomorphism. In particular, $p^{-1}(W)$ is properly $\bigvee$-definably complete iff W is properly $\bigvee$-definably complete.

Lemma 4.4 *Let* $p : Y \longrightarrow X$ *be a strictly properly $\bigvee$-definable covering map. Suppose that* $f, g : Z \longrightarrow Y$ *are continuous strictly properly $\bigvee$-definable maps such that* $p \circ f = p \circ g$. *If* $f(z) = g(z)$ *for some point* $z \in Z$, *then* $f = g$.

Proof. Let $W := \{w \in Z : f(w) = g(w)\}$. Then W and $Z \setminus W$ are properly $\bigvee$-definable subsets of Z (since f and g are strictly properly $\bigvee$-definable maps. By properly $\bigvee$-definable connectedness of Z its enough to show that W is open and closed in Z. Let $w \in W$ and let U be a p-admissible definable open neighbourhood of $p(f(w)) = p(g(w))$, and let V be the definable sheet over U containing $f(w) = g(w)$. Clearly, $O = f^{-1}(V) \cap g^{-1}(V) \cap Z_k$ for Z_k such that $w \in Z_k$, is a definable open neighbourhood of w in Z. We claim that $O \subseteq W$. If $v \in O$, then $f(v), g(v) \in V$ and also $p(f(v)) = p(g(v))$, but $p_{|V}$ is a definable homeomorphism and so $f(v) = g(v)$. Therefore, W is open. Since Z is Hausdorff, W is closed. This assumption on Z is not necessary: if $w \in Z \setminus W$, let V be a definable p-admissible open neighbourhood of $p(f(w))$. If both $f(w)$ and $g(w)$ lie in the same sheet over V, then the argument above shows that $p(f(w)) = p(g(w))$. Therefore, $f(w) \in S$ and $g(w) \in S'$, where S, S' are distinct sheets. But $U = f^{-1}(S) \cap$

$g^{-1}(S') \cap Z_k$ is a definable open neighbourhood of w such that $U \subseteq Z \setminus W$. Therefore, $Z \setminus W$ is open as well. □

Recall that some times we identify definable homotopies under the equivalence $\simeq$ of having a common refinement.

Proposition 4.5 *Let $p : Y \longrightarrow X$ be a strictly properly $\bigvee$-definable covering map. (1) If Γ a definable path in X and $y \in Y$ is such that $p(y) = \inf \Gamma$, then there is a unique definable path $\overline{\Gamma}$ in Y such that $y = \inf \overline{\Gamma}$ and $p \circ \overline{\Gamma} = \Gamma$.*
(2) Suppose that $\Gamma \sim_H \Sigma$ is a definable homotopy of definable paths in X. Let $\overline{\Gamma}$ be a definable lifting of Γ, then there is a unique definable lifting $\overline{H}$ of H (i.e., $p \circ \overline{H} = H$) such that $\overline{\Gamma} \sim_{\overline{H}} \overline{\Sigma}$ where $\overline{\Sigma}$ is a definable lifting of Σ.

Proof. Let $\{U_l : l \in L\}$ a p-admissible family of definable neighbourhoods. (1) Since for each $i \in I$, there is a finite subset L_i of L such that $X_i \subseteq \cup\{U_l : l \in L_i\}$ and $|\Gamma|$ is a definable subset of X, there is a finite subset I' of I such that $|\Gamma| \subseteq \cup\{X_i : i \in I'\}$ and so $|\Gamma| \subseteq \cup\{U_l : l \in \cup\{L_i : i \in I'\}\}$. Therefore $\Gamma \simeq \Gamma_1 \cdots \cdots \Gamma_n$ for some definable paths Γ_j $(j = 1, \ldots, n)$ such that for each $j = 1, \ldots, n$ there is $j(l) \in \cup\{L_i : i \in I'\}$ such that $|\Gamma_j| \subseteq U_{j(l)}$. Since the result clearly holds for each Γ_j the result holds for Γ. (2) This is proved in a similar way, by taking a refinement of H compatible with $\{U_l : l \in L'\}$ where L' is a finite subset of L such that $|H| \subset \cup\{U_l : l \in L'\}$. □

Notation: Referring to proposition 4.5, we denote by $y * \Gamma$ the final point $\sup \overline{\Gamma}$ of the definable lifting $\overline{\Gamma}$ of Γ with initial point $\inf \overline{\Gamma} = y$.

Corollary 4.6 *Let $((Y, y), p, (X, x))$ be a strictly properly $\bigvee$-definable covering space. Then (1) the induced homomorphism $p_* : \pi_1(Y, y) \longrightarrow \pi_1(X, x)$ is injective; (2) if Σ is a definable loop at x, then $y = y * \Sigma$ iff $[\Sigma] \in p_*(\pi_1(Y, y))$ and if Λ and Λ' are two definable paths in X from x to x', then $y * \Lambda = y * \Lambda'$ iff $[\Lambda \cdot \Lambda'^{-1}] \in p_*(\pi_1(Y, y))$.*

Proof. (1) Let Σ be a definable loop at y such that $p_*([\Sigma]) = 1$. And let H be the definable homotopy from $p \circ \Sigma$ to ϵ_x. By definable homotopy lifting, H lifts to a definable homotopy $\overline{H}$ from Σ to some definable path. By uniqueness of $\overline{H}$ its easy to see that this definable path is ϵ_y, and $[\Sigma] = 1$ as required. A similar argument shows (2). □

Proposition 4.7 *Let $p : (Y, y) \longrightarrow (X, x)$ be a strictly properly $\bigvee$-definable covering map. (1) The map $\rho : \pi_1(X, x) \times p^{-1}(x) \longrightarrow p^{-1}(x)$ given by $\rho([\Sigma], u) := u * \Sigma$ is a transitive action, the stabiliser of $u \in p^{-1}(x)$ is $p_*(\pi_1(Y, u))$ and $|p^{-1}(x)| = [\pi_1(X, x) : p_*(\pi_1(Y, u))]$.*

(2) For all $x_1, x_2 \in X$, $|p^{-1}(x_1)| = |p^{-1}(x_2)|$, for all $u, v \in p^{-1}(x)$, $p_*(\pi_1(Y, u))$ and $p_*(\pi_1(Y, v))$ are conjugate subgroups of $\pi_1(X, x)$ and if S is a subgroup of $\pi_1(X, x)$ that is conjugate to $p_*(\pi_1(Y, u))$ then there is $v \in p^{-1}(x)$ such that $S = p_*(\pi_1(Y, v))$.

Proof. (1) Note that by proposition 4.5 (2), ρ is well defined. We have $\rho([\epsilon_x], u) = u$ because the lifting $\overline{\epsilon_x}$ of ϵ_x at u is ϵ_u. Now suppose that $[\Lambda] \in \pi_1(X, x)$. Let $\overline{\Sigma}$ be the definable lifting of Σ at u and let $\overline{\Lambda}$ be the definable lifting of Λ at $u * \Sigma$, then $\overline{\Sigma} \cdot \overline{\Lambda}$ is the definable lifting of $\Sigma \cdot \Lambda$ that begins at u and ends at $(u*\Sigma)*\Lambda$. Therefore, $\rho([\Sigma][\Lambda], u) = \rho([\Sigma \cdot \Lambda], u) = \rho([\Lambda], \rho([\Sigma], u))$. Let $u, v \in p^{-1}(x)$. Since Y is definably path connected, there is a definable path Γ in Y from u to v. $p \circ \Gamma$ is a loop at x whose lifting at x is Γ. Thus $[p \circ \Gamma] \in \pi_1(X, x)$, and $u * (p \circ \Gamma) = v$ i.e., $\rho([p \circ \Gamma], u) = v$ and ρ is transitive.

Let $[\Sigma] \in \pi_1(X, x)_u$ (the stabilizer of u) and let $\overline{\Sigma}$ be the definable lifting of Σ at u. Then $u = u * \Sigma = \sup \overline{\Sigma}$ and so $[\overline{\Sigma}] \in \pi_1(Y, u)$, $[\Sigma] = [p \circ \overline{\Sigma}] \in p_*(\pi_1(Y, u))$. For the reverse inclusion, suppose that $[\Sigma] = [p \circ \Gamma]$ for some $[\Gamma] \in \pi_1(Y, u)$. Then $\overline{\Sigma} = \Gamma$. Therefore, $u * \Sigma = \sup \overline{\Sigma} = \sup \Gamma = u$, and $[\Sigma] \in \pi_1(X, x)_u$. The rest of (1) follows from the theory of G-sets.

(2) We have the following commutative diagram

$$
\begin{array}{ccc}
\pi_1(Y, u_1) & \overset{A}{\rightarrow} & \pi_1(Y, u_2) \\
\downarrow{p_*} & & \downarrow{p_*} \\
\pi_1(X, x_1) & \overset{a}{\rightarrow} & \pi_1(X, x_2)
\end{array}
$$

where $A([\Gamma]) := [\Theta^{-1} \cdot \Gamma \cdot \Theta]$, $a([\Sigma]) := [(p \circ \Theta)^{-1} \cdot \Sigma \cdot (p \circ \Theta)]$ and Θ is a definable path from u_1 to u_2. Since A and a are isomorphisms and p_* is a monomorphism it follows from (1) that $|p^{-1}(x_1)| = |p^{-1}(x_2)|$. The same diagram applied to the case $x_1 = x_2 = x$ shows that for all $u, v \in p^{-1}(x)$, $p_*(\pi_1(Y, u))$ and $p_*(\pi_1(Y, v))$ are conjugate subgroups of $\pi_1(X, x)$.

Suppose now that $S = [\Sigma^{-1}]p_*(\pi_1(Y, u))[\Sigma]$ for some $[\Sigma] \in \pi_1(X, x)$. Let $\overline{\Sigma}$ be the definable lifting of Σ at u. Note that $v := u * \Sigma \in p^{-1}(x)$. Using the commutative diagram we see that $S = p_*(\pi_1(Y, v))$. $\qquad \square$

34

4.2 Liftings of strictly properly $\bigvee$-definable maps

The results of this subsection are all corollaries of following result on the possibility of lifting strictly properly $\bigvee$-definable maps.

Proposition 4.8 *Let $p : (Y, y) \longrightarrow (X, x)$ be a strictly properly $\bigvee$-definable covering map and let $f : (Z, z) \longrightarrow (X, x)$ be a continuous strictly properly $\bigvee$-definable map. Then there is a continuous strictly properly $\bigvee$-definable map $\overline{f} : (Z, z) \longrightarrow (Y, y)$ with $p \circ \overline{f} = f$ iff $f_*(\pi_1(Z, z)) \subseteq p_*(\pi_1(Y, y))$. Such strictly properly $\bigvee$-definable lifting $\overline{f}$, when it exists, is unique.*

Proof. The necessity is clear and the uniqueness follows from lemma 4.4. We will now construct $\overline{f}$. For each $i \in I$ choose $z_i \in Z_i$ such that if $z \in Z_i$ then $z = z_i$, and let Γ_i be a definable path in Z from z to z_i. Given $w \in Z_i$, let $\Delta_{z,w}$ be the definable path $\Gamma_i \cdot \Gamma_{i,z_i,w}$ from z to w where, $\Gamma_{i,z_i,w}$ is given by proposition 2.28. Let $\Sigma_{z,w} := f \circ \Delta_{z,w}$ and put $\overline{f}(w) := y * \Sigma_{z,w}$.

If $w \in Z_i \cap Z_j$ then we have another definable path $\Delta'_{z,w}$ from z to w obtained from Z_j. $f \circ (\Delta'_{z,w} \cdot \Delta_{z,w}^{-1}) = \Sigma'_{z,w} \cdot \Sigma_{z,w}^{-1}$ is a definable loop at x. By hypothesis, $[\Sigma'_{z,w} \cdot \Sigma_{z,w}^{-1}] \in p_*(\pi_1(Y, y))$ and by corollary 4.6 (2), $y * \Sigma_{z,w} = y * \Sigma'_{z,w}$ and so $\overline{f}$ is well defined. Note that the same argument shows that $\overline{f}$ does not depend on the choice of the points $z_i \in Z_i$ or of the definable paths Γ_i.

Its easy to see that $\overline{f}$ is a strictly properly $\bigvee$-definable map. Let $\{U_l : l \in L\}$ be the p-admissible family of definable open neighbourhoods in X. Since f is continuous and strictly properly $\bigvee$-definable, $\{f^{-1}(U_l) : l \in L\}$ is an open cover of Z such that, for each Z_i there is a finite subset S_i of L such that $Z_i \subseteq \bigcup_{s \in S_i} f^{-1}(U_s)$. For each i and each $s \in S_i$ let $V_{is} := Z_i \cap f^{-1}(U_s)$ and let $v_{is} \in V_{is}$. Then V_{is} are open definable sets and $Z_i = \bigcup_{s \in S_i} V_{is}$. Moreover, by the argument above we can assume that the properly $\bigvee$-definable system $\Delta_{z,w}$ of definable paths is, for $w \in V_{is}$ of the form $\Gamma_i \cdot \Gamma_{i,z_i,v_{is}} \cdot \Gamma_{i,v_{is},w}$. Clearly, we get a properly $\bigvee$-definable system $\Lambda_{x,f(w)} := f \circ \Delta_{z,w}$ of definable paths in $f(Z)$ which is of the form $(f \circ \Gamma_i) \cdot (f \circ \Gamma_{i,z_i,v_{is}}) \cdot (f \circ \Gamma_{i,v_{is},w})$. From this we see clearly that $\overline{f}_{|V_{is}}$ is definable and the result follows.

We will now show that $\overline{f}$ is continuous at w: let U be a p-admissible definable neighbourhood of $f(w)$ and let V be an open definable set of $p^{-1}(U)$ that is mapped homeomorphically onto U by p and that contains $\overline{f}(w)$. Choose a definably path connected definable neighbourhood W of w so that $f(W) \subseteq U$.

We need to show that $\overline{f}(W) \subseteq V$. For each $w' \in W$ there is a definable path Λ from w to w' in W, and then we can use $\Gamma \cdot \Lambda$ as the definable path from z to w'. The definable lifting of $f \circ (\Gamma \cdot \Lambda) = (f \circ \Gamma) \cdot (f \circ \Lambda)$ is obtained by first lifting $f \circ \Gamma$ and then lifting $f \circ \Lambda$. Since the latter lifting stays in V, this shows that $\overline{f}(W) \subseteq V$. $\qquad\square$

Corollary 4.9 *(1) Let $p : Y \longrightarrow X$ and $q : Z \longrightarrow X$ be strictly properly $\bigvee$- definable covering maps. There is a strictly properly $\bigvee$-definable isomorphism between the strictly properly $\bigvee$-definable covering spaces iff $p_*(\pi_1(Y,y))$ and $q_*(\pi_1(Z,z))$ are conjugate subgroups of $\pi_1(X,x)$ iff $p^{-1}(x)$ and $q^{-1}(x)$ are isomorphic $\pi_1(X,x)$-sets.*

(2) If $p_*(\pi_1(Y,y)) \subseteq q_*(\pi_1(Z,z))$, then there is a unique strictly properly $\bigvee$-definable covering map $r : (Y,y) \longrightarrow (Z,z)$ such that $p \circ r = q$.

Proof. (1) Suppose that $\phi : (Y,y) \longrightarrow (Z,z)$ is a strictly properly $\bigvee$-definable isomorphism. Then $p_*(\pi_1(Y,y)) = q_*(\phi(Y),\phi(y))$ which is conjugate to $q_*(\pi_1(Z,z))$ by proposition 4.7 (2). On the other hand it is clear that $\phi_{|p^{-1}(x)} : p^{-1}(x) \longrightarrow q^{-1}(x)$ is a bijection, for $[\Sigma] \in \pi_1(X,x)$, if $\overline{\Sigma}$ is the definable lifting of Σ at y then $\phi \circ \overline{\Sigma}$ is the definable lifting of Σ at z. And so ϕ induces an isomorphism $\overline{\phi} : p^{-1}(x) \longrightarrow q^{-1}(x)$ of $\pi_1(X,x)$-sets. Conversely, if $p^{-1}(x)$ and $q^{-1}(x)$ are isomorphic $\pi_1(X,x)$-sets then by the theory of G-sets, the stabilisers $p_*(\pi_1(Y,y))$ of y and $q_*(\pi_1(Z,z))$ of z in $\pi_1(X,x)$ are conjugate. By proposition 4.7 (2), there is $y' \in p^{-1}(x)$ such that $q_*(\pi_1(Z,z)) = p_*(\pi_1(Y,y'))$. By proposition 4.8, there is a strictly properly $\bigvee$-definable map $\phi : Y \longrightarrow Z$ (the strictly properly $\bigvee$-definable lifting of $p : (Y,y') \longrightarrow (X,x)$) such that $q \circ \phi = p$. By considering the strictly properly $\bigvee$-definable lifting of $q : (Z,z) \longrightarrow (X,x)$ we see that ϕ is a homeomorphism and so a strictly properly $\bigvee$-definable isomorphism of pointed strictly properly $\bigvee$-definable pointed coverings spaces $((Y,y'),p,(X,x))$ and $((Z,z),q,(X,x))$.

(2) By proposition 4.8, there is a strictly properly $\bigvee$-definable map $h : (Y,y) \longrightarrow (Z,z)$ (the strictly properly $\bigvee$-definable lifting of $p : (Y,y) \longrightarrow (X,x)$) such that $q \circ h = p$. It remains to show that h is a strictly properly $\bigvee$-definable covering map. Let $\{U_l : l \in L\}$ be a p-admissible family of open definably connected definable neighbourhoods in X with $|L| < \aleph_1$ and let $\{V_k : k \in K\}$ be a q-admissible family of open definably connected definable

36

neighbourhoods in X with $|K| < \aleph_1$. For each $l \in L$ there is a finite set $S_l \subseteq K$ such that $U_l \cap V_s \neq \emptyset$ for all $s \in S_l$ and $U_l = \bigcup_{s \in S_l} U_l \cap V_s$. Let $\{W_{lk} : l \in L, k \in S_l\}$ be the family given by $W_{lk} := U_l \cap V_k$ for all $l \in L$ and $k \in S_l$. Then $\{W_{lk} : l \in L, k \in S_l\}$ is simultaneously a p-admissible and q-admissible family of open definable neighbourhoods in X with $|\{(l,k) : l \in L, k \in S_l\}| < \aleph_1$. Since by o-minimality, W_{lk} as only finitely many definably connected components, we can assume without loss of generality that each W_{lk} is definably connected.

Let $\{O_m : m \in M\}$ be the family $\{q^{-1}(W_{lk}) : l \in L, k \in S_k\}$ and $\{O'_n : n \in N\}$ be the family $\{p^{-1}(W_{lk}) : l \in L, k \in S_l\}$. All of O_m's (resp., O'_n) are open definably connected definable neighbourhoods in Z (resp., in Y) with $|M \cup N| < \aleph_1$. We claim that $\{O_m : m \in M\}$ is a h-admissible family of open definably connected definable neighbourhoods in Z: Given O_m, there are $l \in L$ and $k \in S_l$ such that $q_{|O_m} : O_m \longrightarrow W_{lk}$ is a definable homeomorphism. Now let $\{O'_{n'} : n' \in N' \subseteq N\}$ be the subfamily of $\{O_n : n \in N\}$ given by $p^{-1}(W_{lk})$. Then clearly $h^{-1}(O_m) = \{O'_{n'} : n' \in N' \subseteq N\}$ and $h_{|O'_{n'}} : O'_{n'} \longrightarrow O_m$ is a definable homeomorphism. $\qquad\square$

Corollary 4.10 *Let $p : (Y,y) \longrightarrow (X,x)$ be a strictly properly $\bigvee$-definable covering map and consider $p^{-1}(x)$ as a $\pi_1(X,x)$-set. Then we have canonical isomorphisms*

$$Aut(p^{-1}(x)) \simeq Cov(Y/X) \simeq N_\pi(p_*(\pi_1(Y,y)))/p_*(\pi_1(Y,y)),$$

where π denotes $\pi_1(X,x)$.

Proof. The prove that $Aut(p^{-1}(x)) \simeq Cov(Y/X)$ is contained in the prove of corollary 4.9 (1). The rest follows from the theory of G-sets (see lemma 10.26 [ro]). $\qquad\square$

Definition 4.11 A strictly properly $\bigvee$-definable covering map $p : (Y,y) \longrightarrow (X,x)$ is *regular* if $p_*(\pi_1(Y,y))$ is a normal subgroup of $\pi_1(X,x)$. A strictly properly $\bigvee$-definable covering space (Y,p,X) is a *universal strictly properly $\bigvee$-definable covering space* of X if Y is definably simply connected.

Lemma 4.12 *A strictly properly $\bigvee$-definable covering map $p : (Y, y) \longrightarrow (X, x)$ is regular iff $Cov(Y/X)$ acts transitively on $p^{-1}(x)$.*

Proof. Suppose that $p : (Y, y) \longrightarrow (X, x)$ is regular and let $u, v \in p^{-1}(x)$ Then by proposition 4.7 (2), $p_*(\pi_1(Y, u)) = p_*(\pi_1(Y, v))$. By proposition 4.8, there is a strictly properly $\bigvee$-definable homeomorphism $h : (Y, u) \longrightarrow (Y, v)$ such that $p \circ h = p$; thus $h \in Cov(Y/X)$ and $h(u) = v$. Conversely, assume that $Cov(Y/X)$ acts transitively on $p^{-1}(x)$ and let $u, v \in p^{-1}(x)$. Then there is $h \in Cov(Y/X)$ such that $h(u) = v$. But, $p_*(\pi_1(Y, u)) = p_*h_*(\pi_1(Y, u)) = p_*(\pi_1(Y, v))$. By proposition 4.7 (2), $p_*(\pi_1(Y, u))$ is a normal subgroup of $\pi_1(X, x)$. $\qquad\square$

Corollary 4.13 *Suppose that (Y, p, X) is a strictly properly $\bigvee$-definable covering space. Then $p : (Y, y) \longrightarrow (X, x)$ is regular iff*

$$Cov(Y/X) \simeq \pi_1(X, x)/p_*(\pi_1(Y, y))$$

and (Y, p, X) is a strictly universal properly $\bigvee$-definable covering space of X iff

$$Cov(Y/X) \simeq \pi_1(X, x).$$

4.3 Existence of $\bigvee$-definable covering spaces

Throughout this subsection, G will be a subgroup of $\pi_1(X, x)$ with $|G| < \aleph_1$ (it will follow from the main theorem of this subsection that $|\pi_1(X, x)| < \aleph_1$).

Lemma 4.14 *There is a cover $\{U_s : s \in S\}$ of X with $|S| < \aleph_1$ by definable open subsets U_s's such that every definable loop at $x \in U_s$ is definably homotopic to the constant path ϵ_x at x and for each $i \in I$, there is a finite subset S_i of S with $X_i \subseteq \cup\{U_s : s \in S_i\}$. We call such family a good family of open definable neighbourhoods.*

Proof. If X has a properly $\bigvee$-definable cell decomposition K, then for each 0-cell C_s ($s \in S$) of K let U_s be the union of C_s together with all open k-cells C of K such that $C_s \subseteq \overline{C}$. Clearly, U_s is an open definably path connected definable subset, which by remark 3.14 has the property of the lemma. In general, since each X_i has a properly $\bigvee$-definable cell

38

decomposition, by the argument above, there is a finite such cover of each X_i and from this, the result follows. $\square$

Let X_G be the set of equivalence classes $[\Sigma]_G$ of definable paths Σ in X with initial point $\inf \Sigma = x$ under the following equivalence relation: two such definable paths Σ and Λ are equivalent iff $\sup \Sigma = \sup \Lambda$ and $[\Sigma \cdot \Lambda^{-1}] \in G$. Let $x_G := [\epsilon_x]_G$ and define $p_G : X_G \longrightarrow X$ by $p_G([\Sigma]_G) = \sup \Sigma$.

For each $s \in S$ let $u_s \in U_s$ and $s_G := \{[\Sigma]_G : \sup \Sigma = u_s\}$. Consider the family $\{V_{sk} : s \in S, k \in s_G\}$ where each V_{sk} is the set of all $[\Lambda]_G$ such that there are $[\Sigma]_G \in s_G$ and a definable path Γ in U_s with initial point $\inf \Gamma = u_s$ and $\Lambda = \Sigma \cdot \Gamma$.

Lemma 4.15 *For all $s \in S$ and $k \in s_G$, $|s_G|$ and V_{sk} are independent of the choice of $u_s \in U_s$. Given $s' \in S$ then $|s'_G| = |s_G|$ and $U_s \cap U_{s'}$ is non empty iff there are $k \in s_G$ and $l \in s'_G$ such that $V_{sk} \cap V_{s'l}$ is non empty. If $k_1, k_2 \in s_G$ are such that $k_1 \neq k_2$ then $V_{sk_1} \cap V_{sk_2} = \emptyset$.*

Proof. Let $u'_s \in U_s$, $k' = [\Sigma']_G = [\Sigma \cdot \Theta]_G$ where $\inf \Theta = \sup \Sigma$ and $|\Theta| \subseteq U_s$, $V'_{sk'}$ the corresponding V_{sk}, $[\Lambda]_G \in V_{sk}$, and $k = [\Sigma]_G$. Then $[\Lambda]_G = [\Sigma \cdot \Gamma]_G$ where $\inf \Gamma = \sup \Sigma$ and $|\Gamma| \subseteq U_s$. Hence, $[\Lambda]_G = [\Sigma \cdot \Gamma]_G = [(\Sigma \cdot \Theta) \cdot (\Theta^{-1} \cdot \Gamma)]_G = [\Sigma' \cdot (\Theta^{-1} \cdot \Gamma)]_G$; since $\inf(\Theta^{-1} \cdot \Gamma) = \sup \Sigma'$ and $|(\Theta^{-1} \cdot \Gamma)| \subseteq U_s$ we get $[\Lambda]_G \in V'_{sk'}$. The reverse inclusion is similar.

If $U_s \cap U_{s'}$ is non empty then we have $|s_G| = |s'_G|$ since we can take $u_s = u_{s'} \in U_s \cap U_{s'}$. We also have for the same reason that there are $k \in s_G$ and $l \in s'_G$ such that $V_{sk} \cap V_{s'l}$ is non empty. The general case follows because X is definably path connected. If there are $k \in s_G$ and $l \in s'_G$ such that $[\Gamma]_G \in V_{sk} \cap V_{s'l}$, then $p_G([\Gamma]_G) \in U_s \cap U_{s'}$.

Suppose that $[\Gamma]_G \in V_{sk_1} \cap V_{sk_2}$. Let $k_i = [\Sigma_i]_G$ and $[\Gamma]_G = [\Sigma_i \cdot \Gamma_i]_G$ where $\inf \Gamma_i = \sup \Sigma_i$ and $|\Gamma_i| \subseteq U_s$ for $i = 1, 2$. Since, $\sup \Gamma = p_G([\Gamma]_G) \in p_G(V_{sk}) \cap p_G(V_{sl})$ by the argument above, we can assume that $\sup \Sigma_1 = \sup \Sigma_2$. We have $[\Sigma_1 \cdot \Sigma_2^{-1}] = [\Sigma_1 \cdot (\Gamma_1 \cdot \Gamma_2^{-1}) \cdot \Sigma_2^{-1}]$ (since in U_s every definable loop is definably homotopic to a constant path) $= [(\Sigma_1 \cdot \Gamma_1) \cdot (\Sigma_2 \cdot \Gamma_2)^{-1}] \in G$ and so $[\Sigma_1]_G = [\Sigma_2]_G$. $\square$

For each $s \in S$ let $I_s := \{i \in I : U_s \cap X_i \neq \emptyset\}$ and let $\{X_{Gskj} : s \in S, k \in s_G, j \in I_s\}$ be given by $X_{Gskj} := \{[\Lambda]_G \in V_{sk} : \sup \Lambda \in U_s \cap X_j\}$. Let $\phi_{Gskj} : X_{Gskj} \longrightarrow \phi_j(X_j)$ be given by $\phi_{Gskj} := \phi_j \circ p_G$, and let $\mathbf{X}_G := (X_G, X_{Gskj}, \phi_{Gskj})_{s \in S, k \in s_G, j \in I_s}$

Theorem 4.16 *For every subgroup G of $\pi_1(X,x)$ with $|G| < \aleph_1$, $\mathbf{X}_G$ is a properly $\bigvee$-definably connected, properly $\bigvee$-definable manifold. Moreover, $p_G : (X_G, x_G) \longrightarrow (X, x)$ is a strictly properly $\bigvee$-definable covering map such that $G = p_{G*}(\pi_1(X_G, x_G))$.*

Proof. $p_{G_{|V_{sk}}} : V_{sk} \longrightarrow U_s$ is a bijection, because U_s is definably path connected and every definable loop in U_s at u_s is definably homotopic to the constant path ϵ_{u_s}. This shows that $\mathbf{X}_G$ is a properly $\bigvee$-definable manifold. Moreover, for each $s \in S$ and $j \in I_s$, $p_G^{-1}(U_s \cap X_j)$ is the disjoint union of the definable sets X_{Gskj} with $k \in s_G$ and $p_{G_{|X_{Gskj}}} : X_{Gskj} \longrightarrow U_s \cap X_j$ is a bijection. Therefore, p_G is an open, continuous and surjective strictly properly $\bigvee$-definable map.

We now show that X_G is definably path connected (and therefore, by the above, (X_G, p_G, X) is a strictly properly $\bigvee$-definable covering space). Let $u = [\Sigma]_G \in X_G$. We have $\Sigma \simeq \Sigma_1 \cdots \Sigma_n$ and there are for each $l \in \{1, \ldots, n\}$ there is $s(l) \in S$ such that $|\Sigma_l| \subseteq U_{s(l)}$. Considering the definable bijection $p_{G_{|V_{s(l)k(l)}}} : V_{s(l)k(l)} \longrightarrow U_{s(l)}$ where $k(l)$ is such that $x_G \in V_{s(0)k(0)}$, and for each $j = 0, \ldots, l-1$ $V_{s(j)k(j)} \cap V_{s(j+1)k(j+1)} \neq \emptyset$, it follows that there is a definable path $\overline{\Sigma} \simeq \overline{\Sigma_1} \cdots \overline{\Sigma_n}$ from x_G to u such that $p_G \circ \overline{\Sigma} = \Sigma$.

Finally, let us show that $G = p_{G*}(\pi_1(X_G, x_G))$. Let $[\Sigma] \in \pi_1(X, x)$. Then there is a unique definable lifting $\overline{\Sigma}$ of Σ in X_G with $\inf \overline{\Sigma} = x_G$. On the other hand, $[\Sigma] \in p_{G*}(\pi_1(X_G, x_G))$ iff $\inf \overline{\Sigma} = \sup \overline{\Sigma} = x_G$ iff $[\Sigma]_G = [\epsilon_x]_G$ iff $[\Sigma] = [\Sigma \cdot \epsilon_x^{-1}] \in G$. Therefore, $G = p_{G*}(\pi_1(X_G, x_G))$. $\quad\square$

Corollary 4.17 *Every strictly properly $\bigvee$-definable covering space $((Y,y), p, (X,x))$ is strictly properly $\bigvee$-definably isomorphic to a strictly properly $\bigvee$-definable covering space of the form $((X_G, x_G), p_G, (X, x))$. In particular, there is a universal strictly properly $\bigvee$-definable covering space $(\widetilde{X}, p, X)$.*

4.4 Properly $\bigvee$-definable G-coverings spaces

Definition 4.18 Let G be a group with $|G| < \aleph_1$. An action of G on X is a homomorphism $\rho : G \longrightarrow Aut(X)$ induced by a map $\rho : G \times X \longrightarrow X$, where $Aut(X)$ is the group of all strictly properly $\bigvee$-definable homeomorphisms of X. We often use the notation $g(x) = \rho(g, x)$. The *orbit* x/G of $x \in X$ is the subset $\{g(x) : g \in G\}$ of X. X is the disjoint union of the orbits. We denote

by X/G the set of orbits, and $r : X \longrightarrow X/G$ denotes the map that sends a point into its orbit (and so, $r^{-1}(x/G) = x/G$). X/G has a topology such that $r : X \longrightarrow X/G$ is a continuous and open surjective map, but in general its not clear that X/G can be made into a properly $\bigvee$-definable manifold such that $r : X \longrightarrow X/G$ is a strictly (properly) $\bigvee$-definable map.

Definition 4.19 Two strictly properly $\bigvee$-definable covering spaces $(\overline{Y}, p, Y)$ and $(\overline{X}, q, X)$ are *strictly properly $\bigvee$-definably equivalent* if there are strictly properly $\bigvee$-definable homeomorphisms ϕ and ψ making the following diagram commutative:

$$
\begin{array}{ccc}
\overline{Y} & \overset{\phi}{\to} & \overline{X} \\
\downarrow p & & \downarrow q \\
Y & \overset{\psi}{\to} & X.
\end{array}
$$

Lemma 4.20 *Consider the following commutative diagram of strictly properly $\bigvee$-definable covering maps*

$$
\begin{array}{c}
Z \\
\downarrow p \quad \searrow^{r} \\
\quad \quad Y \\
\quad \swarrow^{q} \\
X
\end{array}
$$

where (Z, p, X) and (Z, r, Y) are regular; let $G = Cov(Z/Y)$ and $H = Cov(Z/X)$. Then there is a commutative diagram

$$
\begin{array}{c}
Z \\
\downarrow p' \quad \searrow^{r'} \\
\quad \quad Z/G \\
\quad \swarrow^{q'} \\
Z/H
\end{array}
$$

of strictly properly $\bigvee$-definable covering spaces, each of which is strictly properly $\bigvee$-definably equivalent to the corresponding strictly properly $\bigvee$-definable covering space in the original diagram.

Proof. We first show that $(Z, r', Z/G)$ is a strictly properly $\bigvee$-definable covering space (here r' is the natural map $r' : Z \longrightarrow Z/G$), strictly properly $\bigvee$-definably equivalent to (Z, r, Y). Note that since (Z, r, Y) is regular, $Cov(Z/Y)$ acts transitively on each fibre $r^{-1}(y)$. This can be used to show that the map $\phi : Y \longrightarrow Z/G$ given by $\phi(y) = r'(r^{-1}(y))$ is a well defined bijective map. Since both r' and r are continuous and open surjective maps, it follows that ϕ is a homeomorphism, from this we get the claim, by putting on Z/G the structure of a properly $\bigvee$-definable manifold via ϕ which then becomes a well defined bijective strictly properly $\bigvee$-definable map.

Since $G \subseteq H$ if follows that for every $z \in Z$, we have $z/G \subseteq z/H$. Define $q' : Z/G \longrightarrow Z/H$ by $q'(z/G) = z/H$. It is now easy to verify that q' is the map we are after. $\qquad\square$

Definition 4.21 We say that a strictly properly $\bigvee$-definable covering space (Z, r, Y) is a *strictly properly $\bigvee$-definable G-covering* if $(Z, r', Z/G)$ is a strictly properly $\bigvee$-definable covering space strictly properly $\bigvee$-definably equivalent to (Z, r, Y).

Corollary 4.22 *Let $(\widetilde{X}, p, X)$ be a strictly properly $\bigvee$-definable universal covering space of X. Then every strictly properly $\bigvee$-definable covering space (Y, q, X) is strictly properly $\bigvee$-definably equivalent to $(\widetilde{X}/G, r, X)$ for some subgroup G of $Cov(\widetilde{X}/X) \simeq \pi_1(X, x)$.*

Proof. There is a unique strictly properly $\bigvee$-definable covering space $(\widetilde{X}, s, Y)$ such that $p = q \circ s$. Since $\widetilde{X}$ is definably simply connected, both $(\widetilde{X}, p, X)$ and $(\widetilde{X}, s, Y)$ are regular strictly properly $\bigvee$-definable covering spaces. Therefore, by lemma 4.20, (Y, q, X) is strictly properly $\bigvee$-definably equivalent to $(\widetilde{X}/G, r, X)$ where $G = Cov(\widetilde{X}/Y)$. $\qquad\square$

Corollary 4.23 *Let $(\widetilde{X}, p, X)$ be a strictly properly $\bigvee$-definable universal covering space of X. Denote the family of all strictly properly $\bigvee$-definable covering spaces of X of the form $(\widetilde{X}/G, r, X)$, where G is a subgroup of $Cov(\widetilde{X}/X)$, by $\mathcal{C}$ and denote the family of all subgroups of $Cov(\widetilde{X}/X)$ by $\mathcal{G}$. Then $\Phi : \mathcal{C} \longrightarrow \mathcal{G}$ defined by $(Z, q, X) \longrightarrow Cov(\widetilde{X}/Z)$ and $\Psi : \mathcal{G} \longrightarrow \mathcal{C}$ defined by $G \longrightarrow (\widetilde{X}/G, r, X)$ are bijections inverse to one another.*

Proof. If $G \subseteq Cov(\widetilde{X}/X)$, then $\Phi\Psi(G) = Cov(\widetilde{X}/(\widetilde{X}/G))$. This group consists of all strictly properly $\bigvee$-definable homeomorphisms $h : \widetilde{X} \longrightarrow \widetilde{X}$ such that

$$
\begin{array}{ccc}
\widetilde{X} & & \\
 & \searrow^{h} & \\
\downarrow^{r} & & \widetilde{X} \\
 & \swarrow^{r} & \\
\widetilde{X}/G & &
\end{array}
$$

If $g \in G$ and $\widetilde{x} \in \widetilde{X}$, then $\widetilde{x}/G = g(\widetilde{x})/G$, and so $rg = r$; hence $g \in \Phi\Psi(G)$ and $G \subseteq \Phi\Psi(G)$. For the reverse inclusion, let $h \in \Phi\Psi(G)$, then $rh = h$. If $\widetilde{x} \in \widetilde{X}$, then $\widetilde{x}/G = h(\widetilde{x})/G$ and there exists $g \in G$ such that $g(h(\widetilde{x})) = \widetilde{x}$. Since $g \in \Phi\Psi(G)$, it follows that $gh \in \Phi\Psi(G)$ and by uniqueness, $gh = 1_{\widetilde{X}}$ and $h = g^{-1} \in G$.

Similarly, its also easy to see that $\Psi\Phi$ is the identity. $\qquad\square$

Corollary 4.24 *Let* $(\widetilde{X}, p, X)$ *be a strictly properly* $\bigvee$*-definable universal covering space of* X. *If* G *is a subgroup of* $Cov(\widetilde{X}/X) \simeq \pi_1(X, x)$, *then* $\pi_1(\widetilde{X}/G, \widetilde{x}/G) \simeq G$.

Proof. We have $\pi_1(\widetilde{X}/G, \widetilde{x}/G) \simeq Cov(\widetilde{X}/(\widetilde{X}/G)) \simeq G$. $\qquad\square$

Definition 4.25 We say that G *acts properly on* X if there is a cover $\{U_k : k \in K\}$ of X with $|K| < \aleph_1$ such that each U_k is an open definably connected definable subset and $gU_k \cap U_k = \emptyset$ for all $g \in G\backslash\{1\}$. We call the $\{U_k : k \in K\}$ a G-*admissible family of definable open subsets of* X.

Its easy to see that if $(Z, r, Z/G)$ is a strictly properly $\bigvee$-definable G-covering space such that G acts on X without fixed points (e.g., $(Z, r, Z/G)$ is regular) then G acts properly on Z.

Corollary 4.26 *Suppose that* G *acts properly on* X. *Then* $(X, r, X/G)$ *is a regular strictly properly* $\bigvee$*-definable covering space and*

$$
G \simeq Cov(X/(X/G)) \simeq \pi_1(X/G, x/G)/r_*(\pi_1(X, x)).
$$

Proof. The natural map $r : X \longrightarrow X/G$ is a continuous, surjective open map with $r^{-1}(r(U)) = \bigcup_{g \in G} gU$ for any definable open subset U of X. Let $\{U_k : k \in K\}$ be a G-admissible family of definable open subsets of X. We claim that there is a structure of a $\bigvee$-definable manifold on X/G such that $\{U_k : k \in K\}$ is an r-admissible family definable open subsets of X/G. Let $k \in K$, if g, h are distinct elements of G then $gU_k \cap hU_k = \emptyset$. If $u \in U_k$, then $u/G = (gu)/G$ for every $g \in G$, and so $r(gu) = r(u)$; hence $r_{|gU_k}$ is surjective. If $r(gu) = r(gv)$ for $u, v \in U_k$, then there is $h \in G$ with $gu = hgv$; hence $gU_k \cap hgU_k \neq \emptyset$, a contradiction. Therefore, $r_{|gU_k}$ is a bijection. Let $Y := X/G$, for each $i \in I$ and $k \in K$, let $Y_{i,k} := r_{|U_k \cap X_i}(U_k \cap X_i)$ and let $\psi_{i,k} : Y_{i,k} \longrightarrow \phi_i(X_i)$ be given by $\psi_{i,k}(y) := \phi_i((r_{|U_k})^{-1}(y))$. Its now easy to see that $(Y, (Y_{i,k}, \psi_{i,k})_{i \in I, k \in K})$ is a (properly) $\bigvee$-definable manifold and (X, r, Y) is a strictly properly $\bigvee$-definable G-covering space with r-admissible family of definable open subsets given by $\{Y_{i,k} : i \in I, k \in K\}$.

Its easy to see that $Cov(X/(X/G))$ acts transitively on a fiber, therefore $(X, r, X/G)$ is regular. The rest follows from previous results. $\qquad\square$

We denote by $G\text{-}COV(X)$ (resp., $G\text{-}COV(X, x)$) the set of equivalence classes of strictly properly $\bigvee$-definable G-coverings (Y, p, X) of X (resp., pointed G-coverings $((Y, y), p, (X, x))$ of (X, x)) under strictly properly $\bigvee$-definable isomorphisms of strictly properly $\bigvee$-definable G-coverings (resp., pointed G-coverings) of X (resp., of (X, x)).

Proposition 4.27 *There is a canonical bijection between $G\text{-}COV(X, x)$ and $Hom(\pi_1(X, x), G)$.*

Proof. We first define a map $A : Hom(\pi_1(X, x), G) \longrightarrow G\text{-}COV(X, x)$ has follows: Let $\rho \in Hom(\pi_1(X, x), G)$ and let $((\widetilde{X}, \widetilde{x}), p, (X, x))$ be the universal strictly properly $\bigvee$-definable covering space of (X, x). Consider the action of $\pi_1(X, x)$ on $\widetilde{X} \times G$ given by $[\Sigma](z, g) := (z * \Sigma, g\rho([\Sigma]^{-1}))$. We claim that $\pi_1(X, x)$ acts properly on $\widetilde{X} \times G$. In fact, if $\{U_l : l \in L\}$ is a p-admissible family of definable open neighbourhoods then for each $l \in L$, there is a strictly properly $\bigvee$-definable homeomorphism $M_l : p^{-1}(U_l) \longrightarrow U_l \times \pi_1(X, x)$. Let $\{V_{l,m} : l \in L, m \in \pi_1(X, x)\} = \{V \subseteq p^{-1}(U_l) : l \in L, M_l(V) = U_l \times \{m\}\}$ and for $l \in L$, $g \in G$ and $m \in \pi_1(X, x)$, let $W_{l,m,g} := V_{l,m} \times \{g\}$. Then, $\{W_{l,m,g} : l \in L, m \in \pi_1(X, x), g \in G\}$ is a cover of $\widetilde{X} \times G$ by open definably connected, definable subsets such that $[\Sigma]W_{l,m,g} \cap W_{l,m,g} = \emptyset$ for all

$[\Sigma] \in \pi_1(X, x) \setminus \{1\}$. Let $(Y, y) := (\widetilde{X} \times G / \pi_1(X, x), (\widetilde{x}, 1) / \pi_1(X, x))$ and let $r : Y \longrightarrow X$ be the strictly properly $\bigvee$-definable map induced by p. There is a natural action of G on Y induced by the natural action of G on $\widetilde{X} \times G$. Its now easy to see, using an argument similar to the one above, that G acts properly on Y and $((Y, y), r, (X, x))$ is a strictly properly $\bigvee$-definable G-covering space.

Let $B : G\text{-}COV(X, x) \longrightarrow Hom(\pi_1(X, x), G)$ be the map defined as follows: given a strictly properly $\bigvee$-definable G covering space $((Y, y), p, (X, x))$, let $\rho \in Hom(\pi_1(X, x), G)$ be determined by $\rho([\Sigma])y = y * \Sigma$. Now, some easy computations, show that the map A is the inverse of the map B. $\qquad\square$

4.5 The Seifert-van Kampen theorem

We give here the proof of the o-minimal version of the Seifert-van Kampen theorem for X without assuming that X is properly $\bigvee$-definably complete. The prove we present here is analogue to Grothendieck proof in the classical case (see [ro]).

Lemma 4.28 *Let $\{X^\alpha : \alpha \in A\}$ with $|A| < \aleph_1$ be a cover of X by properly $\bigvee$-definable open sets with properly $\bigvee$-definable covering maps $p_\alpha : Y^\alpha \longrightarrow X^\alpha$. Suppose that for each $\alpha, \beta \in A$, we have strictly properly $\bigvee$-definable isomorphisms*

$$\theta_{\alpha,\beta} : p_\alpha^{-1}(X^\alpha \cap X^\beta) \longrightarrow p_\beta^{-1}(X^\alpha \cap X^\beta)$$

of strictly properly $\bigvee$-definable coverings of $X^\alpha \cap X^\beta$, such that for every $\alpha, \beta, \gamma \in A$ we have $\theta_{\alpha,\alpha} = 1_{Y^\alpha}$ and $\theta_{\gamma,\alpha} = \theta_{\gamma,\beta} \circ \theta_{\beta,\alpha}$ on $p_\alpha^{-1}(X^\alpha \cap X^\beta \cap X^\gamma)$. Then there is a strictly properly $\bigvee$-definable covering map $p : Y \longrightarrow X$ and there are strictly properly $\bigvee$-definable isomorphisms $\lambda_\alpha : Y^\alpha \longrightarrow p^{-1}(X^\alpha)$ of strictly properly $\bigvee$-definable coverings of X^α such that $\theta_{\alpha,\beta} = \lambda_\beta^{-1} \circ \lambda_\alpha$ on $p_\alpha^{-1}(X^\alpha \cap X^\beta)$ and $Y = \bigcup \{\lambda_\alpha(Y^\alpha) : \alpha \in A\}$.

Moreover, if each $p_\alpha : Y^\alpha \longrightarrow X^\alpha$ is a strictly properly $\bigvee$-definable G-covering and each $\theta_{\alpha,\beta}$ is a strictly properly $\bigvee$-definable isomorphism of strictly properly $\bigvee$-definable G-coverings, then $p : Y \longrightarrow X$ is a strictly properly $\bigvee$-definable G-covering and each $\lambda_\alpha : Y^\alpha \longrightarrow p^{-1}(X^\alpha)$ is a strictly properly $\bigvee$-definable isomorphism of $\bigvee$-definable G-coverings.

45

Proof. We take Y to be the disjoint union of copies Y_α's of the Y^α's, we take on Y the natural charts induced by those of the Y^α's together with the charts induced by the $\theta_{\alpha,\beta}$'s. $\lambda^\alpha : Y^\alpha \longrightarrow Y_\alpha$ is the natural strictly properly $\bigvee$-definable homeomorphism, $p : Y \longrightarrow X$ is defined by $p \circ \lambda_\alpha = p_\alpha$ on Y^α. Since λ_α is a strictly properly $\bigvee$-definable homeomorphism, p is in fact a strictly properly $\bigvee$-definable covering map.

If each $p_\alpha : Y^\alpha \longrightarrow X^\alpha$ is a strictly properly $\bigvee$-definable covering map and each $\theta_{\alpha,\beta}$ is a strictly properly $\bigvee$-definable isomorphism of strictly properly $\bigvee$-definable G-coverings, then there is a unique action of G on Y commuting with each λ_α i.e., $\lambda_\alpha(gy^\alpha) = g\lambda_\alpha(y^\alpha)$ for $g \in G$ and $y^\alpha \in Y^\alpha$. This gives the strictly properly $\bigvee$-definable covering the structure of a strictly properly $\bigvee$-definable G-covering, so that each λ_α is a strictly properly $\bigvee$-definable isomorphism of strictly properly $\bigvee$-definable G-coverings. □

Theorem 4.29 *Let $\{X^\alpha : \alpha \in A\}$ with $|A| < \aleph_1$ be a cover of X by open properly $\bigvee$-definably connected properly $\bigvee$-definable subsets such that for any $\alpha, \beta \in A$ there is $\gamma \in A$ with $X^\gamma = X^\alpha \cap X^\beta$ and for all $\alpha \in A$, $x_0 \in X^\alpha$ where x_0 is some point of X. Then for any group G and any homomorphisms $h_\alpha : \pi_1(X^\alpha, x_0) \longrightarrow G$ such that $h_\beta = h_\alpha \circ i_{\alpha\beta}$ whenever $X^\beta \subseteq X^\alpha$ and $i_{\alpha\beta} : \pi_1(X^\beta, x_0) \longrightarrow \pi_1(X^\alpha, x_0)$ is the homomorphism induced by the inclusion, there is a unique homomorphism $h : \pi_1(X, x_0) \longrightarrow G$ such that $h_\alpha = h \circ j_\alpha$ for all $\alpha \in A$, where $j_\alpha : \pi_1(X^\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ is the homomorphism induced by the inclusion.*

Proof. By proposition 4.27, the homomorphisms $h_\alpha : \pi_1(X^\alpha, x_0) \longrightarrow G$ determine strictly properly $\bigvee$-definable G-coverings $p_\alpha : Y^\alpha \longrightarrow X^\alpha$ together with base points y_0^α over x_0. The equality $h_\beta = h_\alpha \circ i_{\alpha\beta}$ whenever $X^\beta \subseteq X^\alpha$ and $i_{\alpha\beta} : \pi_1(X^\beta, x_0) \longrightarrow \pi_1(X^\alpha, x_0)$ is the homomorphism induced by the inclusion, allows by lemma 4.28 the construction of a strictly properly $\bigvee$-definable G-covering $p : Y \longrightarrow X$ that restricts to the strictly properly $\bigvee$-definable G-coverings $p_\alpha : Y^\alpha \longrightarrow X^\alpha$. This strictly properly $\bigvee$-definable G-covering corresponds to a homomorphism $h : \pi_1(X, x_0) \longrightarrow G$, and the fact that the restricted coverings agree means precisely that $h \circ j_\alpha = h_\alpha$. □

Corollary 4.30 *Suppose that X is properly $\bigvee$-definably complete. Then $\pi_1(X, x)$ is invariant under taking elementary extensions, elementary substructures of $\mathcal{N}$ (containing the parameters over which $\mathbf{X}$ is defined) and under taking expansions of $\mathcal{N}$ and reducts of $\mathcal{N}$ on which $\mathbf{X}$ is defined and X has definable choice.*

Proof. Since X is properly $\bigvee$-definably complete, there is a cover $\{X^\alpha : \alpha \in A\}$ (with $|A| < \aleph_1$) of X by open properly $\bigvee$-definably connected, locally finite properly $\bigvee$-definable subsets such that for any $\alpha, \beta \in A$ there is $\gamma \in A$ with $X^\gamma = X^\alpha \cap X^\beta$ and for all $\alpha \in A$, $x_0 \in X^\alpha$ where x_0 is some point of X. The result now follows from theorem 3.15 and theorem 4.29. $\qquad\square$

4.6 Strictly properly $\bigvee$-definable Čech cohomology

Definition 4.31 A *properly $\bigvee$-definable cover* of X is a covering $\mathcal{U} = \{U_l : l \in L\}$ of X with $|L| < \aleph_1$ such that for each $l \in L$, U_l is an open definably connected subset of X and for each $i \in I$, there is a finite subset L_i of L such that $X_i \subseteq \cup\{U_l : l \in L_i\}$. We say that a properly $\bigvee$-definable cover $\mathcal{U}$ of X is definably simply connected if for each $l \in L$, U_l is definably simply connected.

Definition 4.32 Let G be a group with $|G| < \aleph_1$ and consider G as a properly $\bigvee$-definable manifold of dimension zero. Let $\mathcal{U} = \{U_l : l \in L\}$ be a properly $\bigvee$-definable cover of X. Let $S = \{(m, n) \in L \times L : U_m \cap U_n \neq \emptyset\}$. A *strictly properly $\bigvee$-definable Čech cochain of* $\mathcal{U}$ if a collection $\{g_{m,n} : (m, n) \in S\}$ (denoted $\{g_{m,n}\}$) of strictly properly $\bigvee$-definable continuous maps $g_{m,n} : U_m \cap U_n \longrightarrow G$. A strictly properly $\bigvee$-definable Čech cochain $\{g_{m,n}\}$ of $\mathcal{U}$ is a *strictly properly $\bigvee$-definable Čech cocycle of* $\mathcal{U}$, if for all $m, n, k \in L$ such that $(m, n), (m, k), (k, n) \in S$ the following properties hold: (1) $g_{m,m} = 1_G$; (2) $g_{m,n} = (g_{n,m})^{-1}$ and (3) $g_{m,n} = g_{m,k} g_{k,n}$ on $U_m \cap U_k \cap U_m$. Two strictly properly $\bigvee$-definable Čech cocycles $\{g_{m,n}\}$ and $\{h_{m,n}\}$ of $\mathcal{U}$ are said to be *strictly properly $\bigvee$-definably cohomologous* if there are strictly properly $\bigvee$-definable continuous maps $k_m : U_m \longrightarrow G$ such that $h_{m,n} = (k_m)^{-1} g_{m,n} k_n$ on $U_m \cap U_n$ for all $m, n \in L$ such that $(m, n) \in S$. This is an equivalence relation, the equivalence classes are called *strictly properly $\bigvee$-definable Čech*

cohomology classes on $\mathcal{U}$ with coeficents on G and the set of equivalence classes is denoted by $H^1(X, \mathcal{U}; G)$.

Proposition 4.33 *Suppose that $\mathcal{U}$ is a definably simply connected, properly $\bigvee$-definable cover of X. Then there are canonical bijections between the following sets: $H^1(X, \mathcal{U}; G)$, G-$COV(X)$ and $Hom(\pi_1(X, x), G)/conjugancy$.*

Proof. A canonical bijection between $Hom(\pi_1(X, x), G)/conjugancy$ and G-$COV(X)$ is obtained from the proof of proposition 4.27. We now construct a bijection between $H^1(X, \mathcal{U}; G)$ and G-$COV(X)$.

Let (Y, p, X) be a strictly properly $\bigvee$-definable G-covering. Since each U_l is definably simply connected, there are strictly properly $\bigvee$-definable isomorphisms $\alpha_l : U_l \times G \longrightarrow p^{-1}(U_l)$ of strictly properly $\bigvee$-definable G-coverings. For each $(m, n) \in S$, let $\alpha_{m,n} : U_m \cap U_n \times G \longrightarrow U_m \cap U_n \times G$ be given by $\alpha_{m,n} := (\alpha_m)^{-1} \circ \alpha_n$. Its easy that, for each $(m, n) \in S$, there is a unique strictly properly $\bigvee$-definable continuous map $g_{m,n} : U_m \cap U_n \longrightarrow G$ such that $\alpha_{m,n}(x, g) = (x, gg_{m,n}(x))$. It folows from the definitions that $\{g_{m,n}\}$ is a strictly properly $\bigvee$-definable Čech cocycle on $\mathcal{U}$. Moreover, an argument similar to the one above, shows that the class of $\{g_{m,n}\}$ in $H^1(X, \mathcal{U}; G)$ does not depend on the choice of $\{\alpha_l : l \in L\}$ and depends only on the class of (Y, p, X) in G-$COV(X)$.

Conversely, given a strictly properly $\bigvee$-definable Čech cocycle $\{g_{m,n}\}$ on $\mathcal{U}$ with coeficents in G, the result follows from lemma 4.28 if we take, for each $l, m, n \in L$ such that $(m, n) \in S$, $X^l = U_l$, $Y^l = U_l \times G$, $p_l : Y^l \longrightarrow X^l$ given by $p_l(x, g) := x$ and $\theta_{m,n} : p_m^{-1}(X^m \cap X^n) \longrightarrow p_m^{-1}(X^m \cap X^n)$ given by $\theta_{m,n}(x, g) := (x, gg_{m,n}(x))$. Moreover, its easy to verify that, the strictly properly $\bigvee$-definable G-coverings constructed in this way from strictly properly $\bigvee$-definable cohomologous strictly properly $\bigvee$-definable Čech cocycles are strictly properly $\bigvee$-definably isomorphic G-coverings. $\qquad \square$

Let $H^1((X, x), \mathcal{U}; G)$ be the set of equivalence classes of strictly properly $\bigvee$-definable Čech cocycles $\{g_{m,n}\}$ on $\mathcal{U}$ with coefficients in G such that $g_{m,n}(x) = e$ for all $(m, n) \in S$ with $x \in U_m \cap U_n$, under the equivalence relation given by $\{g_{m,n}\} \sim \{h_{m,n}\}$ iff there are strictly properly $\bigvee$-definable continuous maps $\{k_l : U_l \longrightarrow G : l \in L\}$ such that for all $l, m, n \in L$ with $x \in U_l$ and $(m, n) \in S$, we have $k_l(x) = e$ and $h_{m,n} = (k_m)^{-1} g_{m,n} k_m$.

Corollary 4.34 *Let $\mathcal{U}$ be a definably simply connected properly $\bigvee$-definable cover of X. Then there are canonical bijections between the following sets: $H^1((X,x),\mathcal{U};G)$, $G\text{-}COV(X,x)$ and $Hom(\pi_1(X,x),G)$.*

4.7 Strictly properly $\bigvee$-definable H-groups

Definition 4.35 (X,μ,x_0) is a *strictly properly $\bigvee$-definable H-manifold* if the *strictly properly $\bigvee$-definable multiplication* $\mu : (X \times X, (x_0, x_0) \longrightarrow (X, x_0)$ and the *unit* x_0 are continuous and satisfy $[\mu \circ i_1] \simeq [1_X] = [\mu \circ i_2]$, where $i_1, i_2 : X \longrightarrow X \times X$ are the continuous strictly properly $\bigvee$-definable maps $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$. A strictly properly $\bigvee$-definable *H-group* is (X, μ, ι, x_0) where (X, μ, x_0) is a strictly properly $\bigvee$-definable H-manifold with $[\mu \circ (\mu \times 1_X)] = [\mu \circ (1_X \times \mu)]$ (i.e., μ is strictly properly $\bigvee$-definably H-associative) and the *strictly properly $\bigvee$-definable H-inverse* $\iota : X \longrightarrow X$ is continuous and satisfies $[\mu \circ (\iota \times 1_x) \circ \Delta_X] = [\epsilon_{x_0}] = [\mu \circ (1_x \times \iota) \circ \Delta_X]$ where $\Delta_X : X \longrightarrow X \times X$ is the diagonal map. A strictly properly $\bigvee$-definable H-group (X, μ, ι, x_0) is *strictly properly $\bigvee$-definably H-abelian* if $[\mu] = [\mu \circ \tau]$ where $\tau : X \times X \longrightarrow X \times X$ is given by $\tau(x,y) = (y,x)$.

Definition 4.36 A pointed, continuous strictly properly $\bigvee$-definable map $h : (X, x_0) \longrightarrow (Y, y_0)$ between strictly properly $\bigvee$-definable H-manifolds (X, μ, x_0) and (Y, γ, y_0) (resp., H-groups (X, μ, ι, x_0) and (Y, γ, ζ, y_0)) is called a *strictly properly $\bigvee$-definable H-map* (resp., *H-homomorphism*) if $[h \circ \mu] = [\gamma \circ (h \times h)]$ (resp., also $[h \circ \iota] = [\zeta \circ h]$).

Let Σ and Γ be definable path in X with $\Sigma = \Sigma_1 \cdots \Sigma_k$, $\Gamma = \Gamma_1 \cdots \Gamma_l$ where the Σ_i's (resp., the Γ_j's) are definable basic paths parametrised by $\sigma_i : (a_i, b_i) \longrightarrow X$ (resp., $\gamma_j : (c_j, d_j) \longrightarrow X$). For $t \in [a_i, b_i]$ (resp., $t \in [c_j, d_j]$) we define $(\Sigma, \Gamma)(t) := (\Sigma(t), \Gamma(t))$ to be $(\sigma_i(t), \inf \Gamma)$ (resp., $(\sup \Sigma, \gamma_j(t))$) where, if $\Sigma_i = \epsilon_x$ and so $(a_i, b_i) = \emptyset$ (resp., $\Gamma_j = \epsilon_y$ and so $(c_j, d_j) = \emptyset$) we set $\sigma_i(t) = x$ (resp., $\gamma_j(t) = y$).

Lemma 4.37 *If (X, μ, x_0) is a strictly properly $\bigvee$-definable H-manifold, then $\pi_1(X, x_0)$ is an abelian group.*

Proof. For definable paths Σ and Γ in X let $\Sigma\Gamma$ be the definable path such that $\Sigma\Gamma(t) := \mu(\Sigma(t), \Gamma(t))$. It follows from the definition that: (1)

49

$[\epsilon_{x_0}\Sigma] = [\Sigma\epsilon_{x_0}] = [\Sigma]$; (2) if $[\Sigma] = [\Sigma']$ and $[\Gamma] = [\Gamma']$ then $[\Sigma\Gamma] = [\Sigma'\Gamma']$; and (3) $(\Sigma \cdot \Gamma)(\Sigma' \cdot \Gamma') = (\Sigma\Sigma') \cdot (\Gamma\Gamma')$. Now the lemma follows from the definable homotopies: $[\Sigma \cdot \Gamma] = [(\Sigma\epsilon_{x_0}) \cdot (\epsilon_{x_0}\Gamma)] = [(\Sigma \cdot \epsilon_{x_0})(\epsilon_{x_0} \cdot \Gamma)] = [\Sigma\Gamma]$, $[\Gamma \cdot \Sigma] = [(\epsilon_{x_0}\Gamma) \cdot (\Sigma\epsilon_{x_0})] = [(\epsilon_{x_0} \cdot \Sigma)(\Gamma \cdot \epsilon_{x_0})] = [\Sigma\Gamma]$. $\qquad\square$

Proposition 4.38 *Suppose that $(\overline{X}, p, X)$ is a strictly properly $\bigvee$-definable covering space. Consider the diagram of continuous strictly properly $\bigvee$-definable maps*

$$
\begin{array}{ccc}
Y & \xrightarrow{\overline{f}} & \overline{X} \\
\downarrow^{j} & \nearrow^{\overline{H}} & \downarrow^{p} \\
Y \times |\Gamma| & \xrightarrow{H} & X,
\end{array}
$$

where $j(y) := (y, \inf \Gamma)$ for all $y \in Y$. Then there exists a unique continuous strictly properly $\bigvee$-definable map $\overline{H} : Y \times |\Gamma| \longrightarrow Y$ making the diagram commutative.

Proof. Since Y is properly $\bigvee$-definably connected then so is $Y \times |\Gamma|$, and because $\overline{H}(y, \inf \Gamma) = \overline{f}(y)$ for all $y \in Y$, by lemma 4.4, $\overline{H}$ is unique.

Note that its clearly enough to assume that Γ is a definable basic path parametrised by say $\gamma : (a, b) \longrightarrow Z$. And therefore we can assume as well that $|\Gamma| = [a, b]$. Let $\{U_l : l \in L\}$ be a p-admissible family of open definable subsets of X. Since H is continuous strictly properly $\bigvee$-definable, for each $l \in L$, there is a family $\{V_l^j : j \in J_l\}$ with $|J_l| < \aleph_1$ of open definable subsets of $Y \times I$ such that $\{V_l^j : j \in J_l\} = H^{-1}(U_l)$. Let $\pi_1 : Y \times [a, b] \longrightarrow Y$ and $\pi_2 : Y \times [a, b] \longrightarrow [a, b]$ be the natural projections.

We will first construct $\overline{H}_l^j : V_l^j \longrightarrow \overline{X}$, for each $l \in L$ and $j \in J_l$ such that (1) $p \circ \overline{H}_l^j = H_{|V_l^j}$, (2) for all $(y, a) \in V_l^j$, $\overline{H}_l^j(y, a) = \overline{f}(y)$ and (3) for all $(y, t) \in V_{l'}^{j'} \cap V_l^j$, $\overline{H}_l^j(y, t) = \overline{H}_{l'}^{j'}(y, t)$. Its then clear that under these conditions, the collection $\{\overline{H}_l^j : l \in L, \ j \in J_l\}$ determines the strictly properly $\bigvee$-definable map $\overline{H}$ satisfying the proposition.

Let $\{O_l^n : n \in N_l\}$ be the open definable sheets in $\overline{X}$ over U_l, and let $A_l^n = \overline{f}^{-1}(O_l^n)$ in Y. For each $n \in N_l$, let $V_l^{j,n} := V_l^j \cap \pi_1^{-1}(A_l^n)$. Then $V_l^{j,n}$ is a disjoint cover of V_l^j by open definable sets. Define $\overline{H}_l^{j,n} : V_l^{j,n} \longrightarrow O_l^n$ by $\overline{H}_l^{j,n} := (p_{|O_l^n})^{-1} \circ H$. We have thus constructed $\overline{H}_l^{j,n} : V_l^{j,n} \longrightarrow \overline{X}$,

for each $l \in L$, $j \in J_l$ and $n \in N_l$ such that (1) $p \circ \overline{H}_l^{j,n} = H_{|V_l^{j,n}}$, (2) for all $(y,a) \in V_l^{j,n}$, $\overline{H}_l^{j,n}(y,a) = \overline{f}(y)$ and (3) for all $(y,t) \in V_l^{j,n} \cap V_l^{j',n'}$, $\overline{H}_l^{j,n}(y,t) = \overline{H}_{l'}^{j',n'}(y,t)$.

The collection $\{\overline{H}_l^{j,n} : n \in N_l\}$ clearly determines $\overline{H}_l^j$ satisfying (1), (2) and (3). $\qquad\qquad\square$

Given a strictly properly $\bigvee$-definable H-manifold (X,μ,x_0) (resp., given a strictly properly $\bigvee$-definable H-group (X,μ,ι,x_0)) we define in the obvious way the notion of strictly properly $\bigvee$-definable H-submanifold (resp., strictly properly $\bigvee$-definable H-subgroup, strictly properly $\bigvee$-definable H-centre $Z(X)$ of X, etc.,).

Theorem 4.39 *Suppose that $((Y,y_0),p,(X,x_0))$ is a strictly properly $\bigvee$-definable covering space and (X,μ,x_0) is a strictly properly $\bigvee$-definable H-manifold.*

(1) There is a unique structure (Y,γ,y_0) of a strictly properly $\bigvee$-definable H-manifold on Y such that the diagram below commutes.

$$
\begin{array}{ccc}
Y \times Y & \xrightarrow{\gamma} & Y \\
\downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p} \\
X \times X & \xrightarrow{\mu} & X.
\end{array}
$$

Moreover, if (X,μ,ι,x_0) is a (strictly properly $\bigvee$-definably abelian) strictly properly $\bigvee$-definable H-group then there is a unique structure (Y,γ,ζ,y_0) of a (strictly properly $\bigvee$-definably abelian) strictly properly $\bigvee$-definable H-group such that the following diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\zeta} & Y \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{\iota} & X
\end{array}
$$

is commutative.

(2) With respect to γ, $p^{-1}(x_0)$ is an abelian group isomorphic with

$$
\pi_1(X,x_0)/p_*(\pi_1(Y,y_0)) \simeq Cov(Y/X).
$$

Moreover, if (X,μ,ι,x_0) is a strictly properly $\bigvee$-definable H-group then

$$
\left(p^{-1}(x_0), \gamma_{|p^{-1}(x_0)}, \zeta_{|p^{-1}(x_0)}, y_0\right)
$$

51

is a strictly properly $\bigvee$-definable H-subgroup contained in the strictly properly $\bigvee$-definable H-centre $Z(Y)$ of Y.

Proof. (1) Let $f = \mu \circ (p \times p)$ then by the proof of lemma 4.37 we see that

$$f_*(\pi_1(Y \times Y, (y_0, y_0)) \subseteq p_*(\pi_1(Y, y_0)).$$

Therefore, by proposition 4.8 there is a unique strictly properly $\bigvee$-definable γ such that $\gamma(y_0, y_0) = y_0$ and the diagram in (1) is commutative. To see that γ satisfies the condition of a multiplication of a strictly properly $\bigvee$-definable H-manifold, we consider by proposition 4.38 the strictly properly $\bigvee$-definable liftings $\widetilde{H}_j$ $(j = 1, 2)$ of the corresponding strictly properly $\bigvee$-definable homotopies H_j for $[\mu \circ i_j] = [1_X]$ in the strictly properly $\bigvee$-definable H-manifold X. The rest of (1) is proved in a similar way.

(2) By lemma 4.37, $\pi_1(X, x_0)$ is abelian. Therefore, (Y, p, X) is regular by corollary 4.13 and $Cov(Y/X) \simeq \pi_1(X, x_0)/p_*(\pi_1(Y, y_0))$. We define a bijection $\psi : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$ by $\psi([\Sigma]) = y_0 * \Sigma$. By the diagram in (1), $(p^{-1}(x_0), \gamma_{|p^{-1}(x_0)}, y_0)$ is a strictly properly $\bigvee$-definable H-submanifold. By lemma 4.37 we have, $[\Sigma \cdot \Gamma] = [\Sigma\Gamma]$ and so $y_0 * (\Sigma \cdot \Gamma) = y_0 * (\Sigma\Gamma)$. Therefore, ψ is a homomorphism and from previuos results we see that $ker\psi = p_*(\pi_1(Y, y_0))$. For $y \in Kerp$, the strictly properly $\bigvee$-definable left and right H-translations $L_y, R_y : Y \longrightarrow Y$ given by $L_y(z) = \gamma(z, y)$ and $R_y(z) = \gamma(y, z)$ respectively satisfy $[p \circ L_y] = [p] = [p \circ R_y]$ and $[L_y(y_0)] = [\epsilon_y] = [R_y(y_0)]$, hence $[L_y] = [R_y]$ by proposition 4.38. $\qquad \square$

Similarly we get the following result:

Corollary 4.40 *Suppose that $((Y_i, y_{i,0}), p_i, (X_i, x_{i,0}))$ for $i = 1, 2$ are properly $\bigvee$-definable covering spaces and $(X_i, \mu_i, x_{i,0})$ (resp., $(X_i, \mu_i, \iota_i, x_{i,0})$) are strictly properly $\bigvee$-definable H-manifolds (resp., H-groups) with a strictly properly $\bigvee$-definable H-map (resp., H-homomorphism) $h : (X_1, x_{1,0}) \longrightarrow (X_2, x_{2,0})$. Then there are unique structures $(Y_i, \gamma_i, y_{i,0})$ (resp., $(Y_i, \gamma_i, \zeta_i, y_{i,0})$) of strictly properly $\bigvee$-definable H-manifolds (resp., H-groups) on Y_i and a unique strictly properly $\bigvee$-definable H-map (resp., H-homomorphism) $l : (Y_1, y_{1,0}) \longrightarrow (Y_2, y_{2,0})$ such that the diagram below commutes*

$$
\begin{array}{ccc}
(Y_1, y_{1,0}) & \overset{l}{\to} & (Y_2, y_{2,0}) \\
\downarrow^{p_1} & & \downarrow^{p_2} \\
(X_1, x_{1,0}) & \overset{h}{\to} & (X_2, x_{2,0}).
\end{array}
$$

5 Strictly properly $\bigvee$-definable groups

In this section, $\mathbf{X} = (X, (X_i, \phi_i)_{i \in I})$ and $\mathbf{Y} = (Y, (Y_j, \psi_j)_{j \in J})$ will be (properly) $\bigvee$-definable manifolds and $Z \subseteq X$ will be a (properly) $\bigvee$-definable subset of X. Note that we do not assume that Z, X or Y have definable choice or are (properly) $\bigvee$-definably connected.

5.1 Strictly properly $\bigvee$-definable groups

Definition 5.1 A *strictly (properly) $\bigvee$-definable group* is a group (Z, μ, ι, z_0) on Z with identity z_0 and such that the product map μ and the inverse map ι are strictly (properly) $\bigvee$-definable maps. *Strictly (properly) $\bigvee$-definable rings* are defined in a similar way.

Lemma 5.2 *Suppose that Z is a strictly (properly) $\bigvee$-definable group and let V be a large (properly) $\bigvee$-definable subset of Z. Then countably many translates of V cover Z.*

Proof. This is just like in lemma 2.4 in [p1]: Let $\mathcal{M} \prec \mathcal{N}$ be a small model over which (Z, μ, ι, z_0) and V are defined, and assume without loss of generality that $\mathcal{N}$ is $\aleph_1$-saturated. Let $i \in I^Z$, $a \in Z_i$ and let $c \in Z_i$ be a generic point of Z over M such that $tp(c/Ma)$ is finitely satisfiable in M. Then c is a generic point of Z over Ma (see proof of lemma 2.4 in [p1]). Since V is a large properly $\bigvee$-definable subset of Z, so is $\mu(V, \iota(a))$ and therefore, $c \in \mu(V_k, \iota(a))$ and $a \in \mu(\iota(c), V_k)$ for some $k \in K_i$ where K_i a finite subset of I^V such that $Z_i \subseteq \cup \{\mu(\iota(Z_i), V_k) : k \in K_i\}$ (this exists because μ and ι are strictly (properly) $\bigvee$-definable). Since $tp(c/Ma)$ is finitely satisfiable over M, there is $b \in Z_i(M)$ such that $a \in \mu(b, V_k)$ for some $k \in K_i$. Therefore, by compactness theorem, for each $i \in I^Z$, there are $b_1, \ldots b_{r_i} \in Z_i(M)$ such that for every $a \in Z_i$, $a \in \mu(b_j, V_k)$ for some $j = 1, \ldots, r_i$ and $k \in K_i$. $\square$

Lemma 5.2 together with the properly $\bigvee$-definable cell decomposition theorem, gives similarly to what happens in the definable case (see [p1] and [pps1]) the following result for strictly (properly) $\bigvee$-definable groups and rings. The version of this result (also included in the statement of theorem 5.3) for strictly $\bigvee$-definable groups and rings, which shows that these groups

and rings are strictly $\bigvee$-definable topological groups and rings appears in
[pst2].

Theorem 5.3 *Let Z be a strictly (properly) $\bigvee$-definable group (resp., ring)
and W a strictly (properly) $\bigvee$-definable subgroup (resp., left or right ideal).
Then there are unique structures of properly $\bigvee$-definable manifolds on Z and
W such that Z (resp., W) is a strictly (properly) submanifold of X (resp.,
Z) and the group (resp., ring) operations are continuous (in fact C^p) strictly
properly $\bigvee$-definable maps. Any strictly (properly) $\bigvee$-definable homomor-
phism between strictly (properly) $\bigvee$-definable groups (resp., rings) is a con-
tinuous (in fact C^p) strictly (properly) $\bigvee$-definable homomorphism. More-
over, W is closed in Z, W is open in Z iff $dimW = dimZ$ and, when both
Z and W are strictly properly $\bigvee$-definable groups then $dimW = dimZ$ iff W
has countable index in Z.*

  Proof. For each $i \in I^Z$ we have $Z_i = \cup\{Z_i^s : s \in S_i\}$ where $|S_i| < \aleph_1$
(in fact $|S_i| = 1$ if X is properly $\bigvee$-definable) and each Z_i^s is a definable
subset of X_i. Suppose that $dimZ = k$. Then by cell decomposition theorem,
each Z_i^s is a finite union of cells. For each $i \in I^Z$ and $s \in S_i$, let V_i^s be
the union of the cells of Z_i^s in the cell decomposition of Z_i^s, of dimension
k. Note that each V_i^s is a disjoint union of finitely many definable subsets
$U_{i,1}^s, \ldots, U_{i,n_i}^s$ definably homeomorphic to an open definable subset of N^k and
$V = \cup\{V_i^s : i \in I^Z, \ s \in S_i\}$ is a large open (properly) $\bigvee$-definable subset of
Z. Using the same argument as in the proof of proposition 2.5 in [p1], we can
further assume that: the inverse map is a continuous strictly (properly) $\bigvee$-
definable map from V into V; there is a large (properly) $\bigvee$-definable subset
U of $Z \times Z$ such that U is open and dense in $V \times V$, multiplication is a
continuous strictly (properly) $\bigvee$-definable map from U into V and for any
$a \in V$, if b is a generic of V over a, then $(b, a) \in U$ and $(\iota(b), \mu(b, a)) \in U$.
The rest of the arguments in [p1] show that $U_{i,j}^s$'s give Z the structure of a
properly $\bigvee$-definable manifold such that the group (resp., ring) operations
are continuous (in fact C^p) strictly properly $\bigvee$-definable maps. Lemma 2.6 in
[pst2] shows that the structures of properly $\bigvee$-definable manifolds on Z and
W such that the group (resp., ring) operations are continuous (in fact C^p)
strictly properly $\bigvee$-definable maps and Z (resp., W) is a strictly (properly)
submanifold of X (resp., Z) are unique (i.e., the inclusion maps are strictly
(properly) $\bigvee$-definable homeomorphisms onto their image). The result about

strictly (properly) $\bigvee$-definable homomorphism is proved in lemma 2.8 in [pst2] and lemma 2.6 in [pst2] shows that W is closed in Z and W is open in Z iff $dimW = dimZ$.

Suppose that both W and Z are properly $\bigvee$-definable. Then clearly, if W has countable index in Z the $dimW = dimZ$. Suppose that $dimW = dimZ$. Looking at Z as a properly $\bigvee$-definable manifold and W as an open properly $\bigvee$-definable subset, we see that Z is a countable (disjoint) union of cosets of W in Z. $\qquad\square$

For general strictly $\bigvee$-definable groups, even though we have that, if W has countable index in Z then $dimW = dimZ$ the reciprocal does not hold: take $\mathcal{N} = (N, 0, +, <)$ an $\aleph_1$-saturated extension of $(\mathbb{R}, 0, +, <)$, $Z = (N, 0, +)$ and W the convex hull of $\mathbb{R}$ in Z.

Remark 5.4 We will from on assume that if (Z, μ, ι, z_0) is a strictly (properly) $\bigvee$-definable group (resp., ring), then $\mathbf{Z} = (Z, (Z_k, \tau_k)_{k \in K})$ is the corresponding unique properly $\bigvee$-definable manifold on Z given by theorem 5.3. Since Z is then a properly $\bigvee$-definable subset of Z (relative to $\mathbf{Z}$) we will simply say that Z be a strictly properly $\bigvee$-definable group (resp., ring). By a definable (resp., properly $\bigvee$-definable, $\bigvee$-definable) subset (resp., subgroup, ideal, etc.,) of Z, we will mean a definable (resp., properly $\bigvee$-definable, $\bigvee$-definable) subset (resp., subgroup, ideal, etc.,) of Z relative to $\mathbf{Z}$. Finally, as usual we will some times write xy for $\mu(x, y)$ and x^{-1} for $\iota(x)$.

Corollary 5.5 *Let Z be a strictly properly $\bigvee$-definable group (resp., ring). Then the properly $\bigvee$-definable connected component Z^0 of Z is the smallest strictly properly $\bigvee$-definable subgroup (resp., ideal) of Z of countable index. If $\{Z^s : s \in S\}$ is a decreasing sequence of strictly properly $\bigvee$-definable subgroups (resp., left or right ideals) of Z then $\cap\{Z^s : s \in S\} = \cap\{Z^s : s \in S_0\}$ for some $S_0 \subseteq S$ with $|S_0| < \aleph_1$ and is a strictly properly $\bigvee$-definable subgroup (resp., left or right ideal) of Z.*

Proof. The first part is clear. So suppose that $\{Z^s : s \in S\}$ is a decreasing sequence of strictly properly $\bigvee$-definable subgroups of Z. For each $s \in S$, let $k_s := dimZ^s$. Since $\{k_s : s \in S\} \subseteq \{0, \ldots, dimZ\}$, there are $k_1 < \ldots < k_m$ in $\{0, \ldots, dimZ\}$ and there are disjoint subsets $S_1, \ldots, S_m$ of

S such that $S = S_1 \cup \cdots \cup S_m$ and for each $l \in \{0, \ldots, m\}$, if $s \in S_l$ then $dimZ^s = k_l$. Therefore, since we want to determine $\cap\{Z^s : s \in S\}$, we may assume without loss of generality that, for all $s \in S$, $dimZ^s = r$. It follows from the first part, that for all $s \in S$, the properly $\bigvee$-definable connected component of Z^s is the same properly $\bigvee$-definable subgroup V. Let s_0 be the first element of S (we can assume, without loss of generality that s_0 exists). Then there is a decreasing sequence $\{U_s : s \in S\}$ of properly $\bigvee$-definable countable subsets of Z^{s_0} containing the identity element and such that for each $s \in S$, $Z^s = \cup\{uV : u \in U_s\}$. Note that, there is $S_0 \subseteq S$ such that $|S_0| < \aleph_1$ and $\{U_s : s \in S\} = \{U_s : s \in S_0\}$. Let $U = \cap\{U_s : s \in S\}$. Then U is a countable nonempty (contains the identity) properly $\bigvee$-definable subset of Z^{s_0}. Let $W := \cup\{uV : u \in U\}$. Then W is a strictly properly $\bigvee$-definable subgroup of Z such that $W = \cap\{Z^s : s \in S\}$.

The results for the strictly properly $\bigvee$-definable rings follows from the corresponding results for strictly properly $\bigvee$-definable groups. $\qquad \square$

From (the proof of) corollary 5.5 we easily get the following result.

Corollary 5.6 *Let Z be a strictly properly $\bigvee$-definable group and let $S \subseteq Z$. Then $C_Z(S) = \{z \in Z : \forall s \in S, \ zs = sz\}$, the centraliser of S in Z, is a strictly properly $\bigvee$-definable subgroup. In fact there is $S_0 \subseteq S$ such that $|S_0| < \aleph_1$ and $C_Z(S) = C_Z(S_0)$, and for each $k \in K^{C_Z(S)}$, there is a finite subset A_k of S such that $C_Z(S) \cap Z_k = C_Z(A_k) \cap Z_k$. In particular, if A is a subgroup of Z, then the centre of $C_Z(A)$ is a strictly properly $\bigvee$-definable subgroup of Z containing A, which is normal if A is normal.*

Similarly to corollary 2.15 in [p1] (see also proposition 5.6 in [p2]) we get:

Corollary 5.7 *Let Z be an infinite strictly properly $\bigvee$-definable group. Then Z has an infinite strictly properly $\bigvee$-definable abelian subgroup.*

We finish this subsection with a result that generalises a theorem from [e1] on definable groups.

Theorem 5.8 *Let X be a strictly properly $\bigvee$-definable group and let Z be a normal strictly properly $\bigvee$-definable subgroup of X. Then we have strictly properly $\bigvee$-definable extension $1 \to Z \to X \xrightarrow{j} Y \to 1$ of strictly properly $\bigvee$-definable groups, with strictly properly $\bigvee$-definable section $s : Y \longrightarrow X$.*

Proof. This is proved by adapting the corresponding result from [e1] for definable groups. We first find Y. The argument in [e1] shows that for each $i \in I$, there is a large definable subset U_i of X_i and there are definable functions $l_{i,1}, \ldots, l_{i,m} : U_i \longrightarrow N$ such that for each $x \in U_i$, there is $z \in xV \cap X_i$ with $\phi_i(z) = (l_{i,1}(x), \ldots, l_{i,m}(x))$ and for all $y \in U_i$, if $xV = yV$ then $(l_{i,1}(x), \ldots, l_{i,m}(x)) = (l_{i,1}(y), \ldots, l_{i,m}(y))$. Let $U = \cup\{U_i : i \in I\}$. Then U is a large properly $\bigvee$-definable subset of X. Let $I = \{0, 1, \ldots\}$ be an enumeration of I and let $l : U \longrightarrow X$ the strictly properly $\bigvee$-definably map defined inductively in the following way: for $x \in U_0$, $l(x) := \phi_i^{-1}(l_0(x))$; suppose that l has been defined on $U_0 \cup \cdots \cup U_k$, then we define l on U_{k+1} by $l(x) := \phi_{k+1}^{-1}(l_{k+1}(x))$ if $xV \cap (U_0 \cup \cdots \cup U_k) = \emptyset$ or $l(x) := l(y)$ for some (for all) $y \in xV \cap (U_0 \cup \cdots \cup U_k)$. Clearly, $l : U \longrightarrow X$ is well defined. Since U is a large properly $\bigvee$-definable subset of X, there is a subset $\{x_s : s \in S\}$ of X with $|S| < \aleph_1$ such that $X = \cup\{x_s U : s \in S\}$. Let $S = \{0, 1, \ldots\}$ be an enumeration of S and let $L : X \longrightarrow X$ be the strictly properly $\bigvee$-definable map defined inductively in the following way: for $x \in x_0 U$, $L(x) := x_0 l(x_0^{-1} x)$; suppose that L has been defined on $x_0 U \cup \cdots \cup x_k U$, then we define L on $x_{k+1} U$ by $L(x) := x_{k+1} l(x_{k+1}^{-1} x)$ if $xV \cap (x_0 U \cup \cdots \cup x_k U) = \emptyset$ or $L(x) := L(y)$ for some (for all) $y \in xV \cap (x_0 U \cup \cdots \cup x_k U)$. Clearly, $L : X \longrightarrow X$ is well defined and moreover, for $x, y \in X$, $L(x) = L(y)$ iff $xV = yV$. Take $Y := L(X)$ and $j = L : X \longrightarrow Y$. It turns out that Y is a properly $\bigvee$-definable subset of X with a strictly properly $\bigvee$-definable group structure given by $xy = L(L^{-1}(x) L^{-1}(y))$.

To find the strictly properly $\bigvee$-definable section $s : Y \longrightarrow X$ we use the same argument, just like in the definable case (see [e1]). □

5.2   The centerless case

Definition 5.9 We say that a strictly properly $\bigvee$-definable group X is *properly $\bigvee$-definably semisimple* if X has no strictly properly $\bigvee$-definable normal abelian subgroup of dimension bigger than zero.

In particular, a strictly properly $\bigvee$-definable group has centre of dimension zero. The following lemma will be useful later.

Lemma 5.10 *Let X be a properly $\bigvee$-definably connected strictly properly $\bigvee$-definable group. Then every strictly properly $\bigvee$-definable normal subgroup*

of X of dimension zero is contained in $Z(X)$ and if $Z(X)$ has dimension zero then, $X/Z(X)$ is a centerless strictly properly $\bigvee$-definable group.

Proof. This is proved in the same way as in the definable case (see [e1]).
□

Theorem 5.11 *If X is a centerless, properly $\bigvee$-definably semisimple, properly $\bigvee$-definably connected strictly properly $\bigvee$-definable group, then $X = X_1 \times \cdots \times X_l$ and for each $k \in \{1, \ldots, l\}$, there is a definable real closed field R_k such that there is no definable bijection between a distinct pair among the R_k's, and there is an R_k-semialgebraically connected, R_k-semialgebraic subgroup G_k of $GL(n_k, R_k)$ which is a direct product of R_k-semialgebraically simple, R_k-semialgebraic subgroups of $GL(n_k, R_k)$ such that X_k is strictly properly $\bigvee$-definably isomorphic to a $\bigvee$-definable open and closed subgroup of G_k.*

Proof. The proof is a modification of the corresponding result in [pps1] for definably semisimple groups. We will therefore assume the readers familiarity with the terminology of [pps1].

Arguing as in the proof of theorem 3.1 in [pps1] and using the fact that the centraliser $C_X(U)$ in X of any subset U of X is a strictly properly $\bigvee$-definable subgroup of X (see corollary 5.5), we see that X is a direct product of strictly properly $\bigvee$-definable unidimensional subgroups. So we may assume that X is unidimensional. Further, we may also assume that $\mathcal{N}$ is $\aleph_1$-saturated and just like in [pps1], there is an open transitive interval M such that $e = (d, \ldots, d)$ for some $d \in M$, where $e = \phi_{i_0}(x_0)$, x_0 is the identity of X and $x_0 \in X_{i_0}$ and moreover, if $B = M^n$ where $n = dimX$ then $\phi_{i_0}^{-1}(B)$ is an open definable neighbourhood of x_0. Let $\rho : M \longrightarrow B$ be the continuous injection defined as $\rho(x) = (x, d, \ldots, d)$. Let $M^+ := \{b \in M : b > d\}$, for $b \in M^+$ let $M_b := \{c \in M : d < c < b\}$, and let $Y_b := C_X(\overline{M}_b)$ where $\overline{M}_b := \phi_{i_0}^{-1}(\rho(M_b))$. Clearly, $\{Y_b : b \in M^+\}$ is a sequence of strictly properly $\bigvee$-definable subgroups of X such that if $b' < b$ then $Y_b \subseteq Y_{b'}$ and therefore, by corollary 5.5, $\{C_X(Y_b) : b \in M^+\}$ is a sequence of strictly properly $\bigvee$-definable subgroups of X such that if $b' < b$ then $C_X(Y_{b'}) \subseteq C_X(Y_b)$. Let $Y = \cup\{Y_b : b \in M^+\}$. Then $C_X(Y) = \cap\{C_X(Y_b) : b \in M^+\}$. Hence by corollary 5.5, there is subset $\{b_s : s \in S\}$ of M^+ with $|S| < \aleph_1$ and such that $C_X(Y) = \cap\{C_X(Y_{b_s}) :$

$s \in S\}$. By saturation, there is $b \in M^+$ such that for all $s \in S$, $b > b_s$. Therefore, $C_X(Y) = C_X(Y_b)$ and $Y = Y_b$ (since $\overline{M}_b \subseteq C_X(Y_b) = C_X(Y)$, we have $Y \subseteq C_X(\overline{M}_b) = Y_b$). Since X is centerless and properly $\bigvee$-definably connected, $dimY < dimX$ (otherwise, $Y = X$ and $\overline{M}_b \subseteq Z(X)$). Hence, B cannot be covered by finitely many left cosets of Y, and arguing as in [pps1], there is a definable real closed field R on some open subinterval of M. Furthermore, just like in the proof of theorem 3.2 in [pps1] we see that $Ad : X \longrightarrow GL(n,R)$, where $Ad(x)$ is the differential at x_0 of the strictly properly $\bigvee$-definable automorphism $a(x) : X \longrightarrow X$ given by $a(x)(z) := xzx^{-1}$, is a strictly properly $\bigvee$-definable injective homomorphism. Let $G := Aut(\mathbf{x}) < GL(n,R)$ where $\mathbf{x}$ is the Lie algebra of X. By theorem 2.37 in [pps1], G and G^0 are R-semialgebraically semisimple R-semialgebraic groups of dimension $dimX$, and clearly, by theorem 5.3 $Ad(X)$ is an open and closed $\bigvee$-definable subgroup of G and since X is properly $\bigvee$-definably connected, $Ad(X)$ is an open and closed strictly properly $\bigvee$-definable subgroup of G^0. $\qquad\square$

Corollary 5.12 *Let X be a properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable group such that $dimX = 1$. Then, X is abelian, divisible and either X is torsion-free and properly $\bigvee$-definably ordered or X is a definably compact definable group.*

Proof. Suppose that X is not abelian. Then every strictly properly $\bigvee$-definable subgroup of X has dimension zero. In particular, $dimZ(X) = 0$ and by lemma 5.10, $X/Z(X)$ is a centerless, strictly properly $\bigvee$-definably semisimple strictly properly $\bigvee$-definable group of dimension one. But by theorem 5.11 we get a contradiction. The rest, follows by adapting the proofs of the corresponding results for definable groups (see [r] and [PiS1]). $\qquad\square$

5.3 The solvable case

Recall from [e1] that a definable abelian group U *has no definably compact parts* if there are definable subgroups $1 = U_0 < U_1 < \cdots < U_n = U$ such that for each $j \in \{1, \ldots, n\}$, U_j/U_{j-1} is a one-dimensional definably connected, torsion-free definable group; and a definable solvable group U *has no definably compact parts* if there are definable subgroups $1 = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_n = U$

such that for each $j \in \{1, \ldots, n\}$, U_j/U_{j-1} is a definable abelian group with no definably compact parts. Definable solvable groups with no definably compact parts are classified in [e1]. The next result (theorem 5.13), uses this fact to reduce the classification of strictly properly $\bigvee$-definable solvable groups to the the classification of properly $\bigvee$-definably complete such groups.

Note that, theorem 5.8 makes possible to develop group extension theory and group cohomology theory in the category of strictly properly $\bigvee$-definable groups with strictly properly $\bigvee$-definable homomorphisms, just like in the category of definable groups with definable homomorphisms treated in [e1]. The proof of the next theorem will use this theory, we therefore assume the readers familiarity the corresponding results from [e1].

Theorem 5.13 *If X is a strictly properly $\bigvee$-definable solvable group. Then we have a strictly properly $\bigvee$-definable extension $1 \to Z \to X \to Y \to 1$, where Z is a definable solvable group with no definably compact parts and Y is a properly $\bigvee$-definably complete, strictly properly $\bigvee$-definable solvable group.*

Proof. The proof is just like in the definable case ([e1]) and is based in the main result of [ps]. We therefore, need to show the analogue of the main result of [ps] in the our more general context. Suppose that X is not properly $\bigvee$-definably complete, and let $\sigma : (a, b) \longrightarrow X$ be a definable injective map such that $\lim_{t \longrightarrow b^-} \sigma(t)$ does not exist in X. Let $I := \sigma(a, b)$ with the natural order $<$, for $b \in I$ let $I^{>b} := \{x \in I : x > b\}$, and for each $x \in X$, let $xI := \{xt : t \in I\}$. As in [ps], define a properly $\bigvee$-definable relation $\prec_I$ on X by $x \prec_I y$ iff for all $t \in I$, for all V definable open neighbourhood of y, there $s \in I$ and U a definable open neighbourhood of x such that $UI^{>s} \subseteq VI^{>t}$. And let $\sim_I$ be defined by $x \sim_I y$ iff $x \prec_I y$ and $y \prec_I x$. Arguing just like in [ps], we see that: (i) $\sim_I$ is a properly $\bigvee$-definable equivalence relation on X; (ii) the class X_1 of the identity 1 of X is a strictly properly $\bigvee$-definable subgroup and (iii) the equivalence classes of $\sim_I$ are exactly the left cosets of X_1. Moreover, lemma 3.7 [ps] shows that X_1 is in fact definable with dimension less than or equal to one, lemma 3.8 [ps] shows that X_1 has dimension one and lemma 3.9 [ps] shows that X_1 is torsion-free. $\qquad \square$

5.4 Covers of strictly properly $\bigvee$-definable groups

We are now ready to prove the main results of the paper, theorem 5.15 and theorem 5.16 below. But first we need the following lemma.

Lemma 5.14 *Let X be a strictly properly $\bigvee$-definable group (resp., ring). Then X has strong definable choice.*

Proof. Let $R(X)$ be the maximal, properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable solvable normal subgroup of X. Then, since we have a strictly properly $\bigvee$-definable extension $1 \to R(X) \to X \to Y \to 1$ where Y is properly $\bigvee$-definably semisimple, which by theorem 5.11 has strong definable choice, its enough to show that $R(X)$ has strong definable choice. One the another hand, by theorem 5.13, and a similar argument, its enough to show that a properly $\bigvee$-definably complete, strictly properly $\bigvee$-definable group has strong definable choice. But this can be proved using the same argument used in [e1] to prove that a definably compact definable group has strong definable choice. $\square$

Theorem 5.15 *Let (X, μ, ι, x_0) be a strictly properly $\bigvee$-definable group and suppose that $((Y, y_0), p, (X, x_0))$ is a strictly properly $\bigvee$-definable covering space.*

(1) Then there is a unique structure (Y, γ, ζ, y_0) of a strictly properly $\bigvee$-definable group on Y such that the diagram

$$
\begin{array}{ccc}
Y \times Y & \xrightarrow{\gamma} & Y \\
\downarrow{p \times p} & & \downarrow{p} \\
X \times X & \xrightarrow{\mu} & X.
\end{array}
$$

and the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\zeta} & Y \\
\downarrow{p} & & \downarrow{p} \\
X & \xrightarrow{\iota} & X
\end{array}
$$

are commutative.

(2) Moreover, $(p^{-1}(x_0), \gamma_{|p^{-1}(x_0)}, \zeta_{|p^{-1}(x_0)}, y_0)$ is an abelian strictly properly $\bigvee$-definable subgroup isomorphic with

$$
\pi_1(X, x_0)/p_*(\pi_1(Y, y_0)) \simeq Cov(Y/X)
$$

61

and contained in the centre $Z(Y)$ of Y.

Proof. This result follows immediately from theorem 4.39, but it can also be proved directly using the same argument and proposition 4.8 instead of proposition 4.38. $\square$

Theorem 5.16 *If $((Y_i, y_{i,0}), p_i, (X_i, x_{i,0}))$ are strictly properly $\bigvee$-definable covering spaces, $(X_i, \mu_i, \iota_i, x_{i,0})$ are strictly properly $\bigvee$-definable groups and $h : (X_1, x_{1,0}) \longrightarrow (X_2, x_{2,0})$ is a strictly properly $\bigvee$-definable homomorphism, then there are unique structures $(Y_i, \gamma_i, \zeta_i, y_{i,0})$ of strictly properly $\bigvee$-definable groups on Y_i and there is a unique strictly properly $\bigvee$-definable homomorphism $l : (Y_1, y_{1,0}) \longrightarrow (Y_2, y_{2,0})$ such that the diagram below commutes*

$$\begin{array}{ccc} (Y_1, y_{1,0}) & \xrightarrow{l} & (Y_2, y_{2,0}) \\ \downarrow{p_1} & & \downarrow{p_2} \\ (X_1, x_{1,0}) & \xrightarrow{h} & (X_2, x_{2,0}). \end{array}$$

Proof. This result is proved in a similar way to theorem 5.15. $\square$

Clearly, there are results analogue to theorem 5.15 and theorem 5.16 for strictly properly $\bigvee$-definable rings.

Corollary 5.17 *If (X, p, Z) be a strictly properly $\bigvee$-definable covering space, then X is properly $\bigvee$-definably complete (resp., abelian, nilpotent or solvable) iff Z is properly $\bigvee$-definably complete (resp., abelian, nilpotent or solvable).*

Proof. This is a consequence of theorem 5.15 and corollary 5.5. $\square$

Theorem 5.18 *Let X be properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable group and let Z be a strictly properly $\bigvee$-definable normal subgroup of X with $dim Z = 0$. Then $(X, j, X/Z)$ is a regular strictly properly $\bigvee$-definable Z-covering space, and*

$$Z \simeq Cov(X/(X/Z)) \simeq \pi_1(X/Z, x/Z)/j_*(\pi_1(X, x)).$$

In particular, if $h : Y \longrightarrow X$ is a strictly properly $\bigvee$-definable surjective homomorphism such that $dim(Ker h) = 0$, then (Y, h, X) is a regular strictly properly $\bigvee$-definable $Ker h$-covering space.

Proof. By corollary 4.26, its enough to show that Z acts properly on X. For $i \in I$, let $\sim_i$ be the definable equivalence relation on X_i given by $x \sim_i y$ iff $xZ \cap X_i = yZ \cap X_i$ i.e., iff there is $z \in Z_i$ such that $y = zx$ or $x = zy$ (where Z_i is some finite subset of Z, which exists by o-minimality since every thing is strictly properly $\bigvee$-definable). Clearly, for all $x \in X_i$, $|x/ \sim_i| \leq |Z_i|$ and there are: a positive natural number m_i, disjoint definable subsets $A_{i,l}$ $(l = 1, \ldots, m_i)$ of X_i whose union is X_i, and positive natural numbers $k_{i,1} > k_{i,2} > \cdots > k_{i,m_i}$ such that for all $x \in A_{i,l}$, we have: $|x/ \sim_i| = k_{i,l}$ and $x/ \sim_i \subseteq A_{i,l}$. Therefore $\sim_i$ induces a definable equivalence relation on $A_{i,l}$ by restriction and there are subsets $Z(i,l)$ of Z_i such that for all $x, y \in A_{i,l}$, $x \sim_i y$ iff there is $z \in Z(i,l)$ such that $y = zx$ or $x = zy$. The definable set $\{(x,y) \in A_{i,l} \times X_i : y \in x/ \sim_i\}$ is a disjoint union of the diagonal $\Delta_{i,l}$ of $A_{i,l}$ together with the disjoint definable sets $R_{i,l,z}$ (where $z \in Z(i,l)$). Let $\alpha_i, \beta_i : X_i \times X_i \longrightarrow X_i$ be the definable maps given by $\alpha_i(x,y) = x$ and $\beta_i(x,y) = y$. Then $R_{i,l,z}$ is the graph of the definable homeomorphism $\gamma_{i,l,z} : \alpha_i(R_{i,l,z}) \longrightarrow \beta_i(R_{i,l,z})$ given by $\gamma_{i,l,z}(x) := \beta_i(R_{i,l,z} \cap \{(x,y) : y = zx\})$ or of the definable homeomorphism $\gamma_{i,l,z^{-1}} : \alpha_i(R_{i,l,z}) \longrightarrow \beta_i(R_{i,l,z})$ given by $\gamma_{i,l,z^{-1}}(x) := \beta_i(R_{i,l,z} \cap \{(x,y) : zy = x\})$. Let $A_{i,l,z} := \alpha_i(R_{i,l,z})$ and let $B_{i,l,z} := \beta_i(R_{i,l,z})$. Its clear that there is an open definable subset V_i of X_i which is the interior in X_i of a definable set of the form $A_{i,l_1,z_1} \cup \cdots \cup A_{i,l_r,z_r}$ and such that: (1) for every $x \in X_i \setminus V_i$ there is $z \in Z_i$ such that $x \in \overline{zV_i}$; (2) $dim(X_i \setminus \{zV_i : z \in Z_i\}) < dim X_i$; and (3) for every $z \in Z_i$, if $z \neq 1$ then $zV_i \cap V_i = \emptyset$.

Since the properly $\bigvee$-definable subset $\cup\{zV_i : i \in I, z \in Z_i\}$ is large in X, there is a subset $\{x_s : s \in S\}$ of X with $|S| < \aleph_1$ and such that $X = \cup\{zx_sV_i : i \in I, s \in S, z \in Z_i\}$ (note that by lemma 5.10, $Z \subseteq Z(X)$). To finish we show that $\{zx_sV_i : i \in I, s \in S, z \in Z_i\}$ is a Z-admissible family of definable open subsets of X. In fact, if $u \in Z$ and $uzx_sV_i \cap zx_sV_i \neq \emptyset$, then $uV_i \cap V_i \neq \emptyset$. Therefore, $u \in Z_i$ or $u^{-1} \in Z_i$ and so $u = 1$. $\qquad\square$

5.5   Local strictly properly $\bigvee$-definable isomorphism

Definition 5.19 Let X and Y be strictly properly $\bigvee$-definable groups, U an open properly $\bigvee$-definable neighbourhood of the identity in X, and $f : U \longrightarrow Y$ a strictly properly $\bigvee$-definable map. We say that f is a *locally strictly properly $\bigvee$-definable homomorphism* if for all $x, y \in U$ such that

$xy \in U$, we have $f(xy) = f(x)f(y)$ and there is a open properly $\bigvee$-definable neighbourhood V of the identity of X such that $V^{-1}V \subseteq U$ and $\{x_lV : l \in L\}$ is an open cover of X with $|L| < \aleph_1$, such that for each $i \in I$, there is a finite subset L_i of L with $X_i \subseteq \cup\{x_lV : l \in L_i\}$.

Theorem 5.20 *Let X and Y be strictly properly $\bigvee$-definable groups, U an open properly $\bigvee$-definable neighbourhood of the identity in X, and let $f : U \longrightarrow Y$ be a locally strictly properly $\bigvee$-definable homomorphism. If X is definably simply connected, then f is uniquely extendible to a strictly properly $\bigvee$-definable homomorphism $\overline{f} : X \longrightarrow Y$.*

  Proof. Let $x \in X$ and consider a definable path Γ in X from the identity 1 to x. Then, there is a finite subset L' of L such that $|\Gamma| \subseteq \cup\{x_lV : l \in L'\}$ and therefore, there are definable paths $\Gamma_1, \ldots, \Gamma_m$ in X such that $\Gamma = \Gamma_1 \cdot \cdots \cdot \Gamma_m$ and for each $j \in \{1, \ldots, m\}$, $|\Gamma_j| \subseteq x_{l_j}V$ for some $l_j \in L'$. From this, it follows that for each $j \in \{1, \ldots, m\}$, $|\Gamma_j|^{-1}|\Gamma_j| \subseteq V \subseteq U$. Now define

$$\overline{f}_\Gamma(x) = f((\inf \Gamma_1)^{-1} \sup \Gamma_1) f((\inf \Gamma_2)^{-1} \sup \Gamma_2) \cdots f((\inf \Gamma_m)^{-1} \sup \Gamma_m).$$

The property of f in U shows implies that $\overline{f}_\Gamma(x) = \overline{f}_\Sigma(x)$ for any definable path in X from 1 to x such that $\Gamma \simeq \Sigma$.
  We now show that $\overline{f}_\Gamma(x)$ is determined independently of the choice of the definable path Γ. Let Σ be another definable path from 1 to x. Since X is definably simply connected, there is a definable homotopy $\Gamma \sim_H \Sigma$ between Γ and Σ. Similarly as before, there is a finite subset L'' of L such that $|H| \subseteq \cup\{x_lV : l \in L''\}$ and for every k-cell K $(k = 0, 1, 2)$ of H, $|K| \subseteq x_{l_K}V$ for some $l_K \in L''$. By the property of f in U and by the construction of $\overline{f}_\Gamma(x)$ its enough to show the claim when H is a k-cell $(k = 0, 1, 2)$ such that $|H| \subseteq x_lV$ for some $l \in L$ and $\Gamma \sim_H^0 \Sigma$. But under these assumptions, the claim is clear.
  Now define $\overline{f} : X \longrightarrow Y$, by $\overline{f}(x) := \overline{f}_\Gamma(x)$ for some (for every) definable path Γ in X from 1 to x. By construction, $\overline{f}$ is an extension of f and, if we consider a properly $\bigvee$-definable system of definable paths in X, we see that $\overline{f}$ is a strictly properly $\bigvee$-definable map. Moreover, $\overline{f}$ is continuous because f, the multiplication and inverse map on X are continuous.

Let $x, y \in X$ and let Γ and Σ be definable paths in X from 1 to x and y respectively. Then, $\Gamma\Sigma$ (notation from the proof of lemma 4.37) is a definable path in X from 1 to xy and we have $\overline{f}_\Gamma(x)\overline{f}_\Sigma(y) = \overline{f}_{\Gamma\Sigma}(xy)$. This shows that $\overline{f}(x)\overline{f}(y) = \overline{f}(xy)$. $\qquad\square$

Two strictly properly $\bigvee$-definable groups X and Y are called *locally strictly properly $\bigvee$-definably isomorphic*, if there are locally strictly properly $\bigvee$-definable homomorphisms $f : U \subseteq X \longrightarrow Y$ and $g : V \subseteq Y \longrightarrow X$ such that $g \circ f_{|f^{-1}(f(U)\cap V)} = 1_{f^{-1}(f(U)\cap V)}$ and $f \circ g_{|g^{-1}(g(V)\cap U)} = 1_{g^{-1}(g(V)\cap U)}$. Note that, in a strictly properly $\bigvee$-definable covering space (Y, p, X), Y and X are locally strictly properly $\bigvee$-definably isomorphic.

Corollary 5.21 *If the strictly properly $\bigvee$-definable groups X and Y are definably simply connected, then X and Y are locally strictly properly $\bigvee$-definably isomorphic iff X and Y are strictly properly $\bigvee$-definably isomorphic.*

Proof. Let $f : U \subseteq X \longrightarrow Y$ and $g : V \subseteq Y \longrightarrow X$ be as in the definition of locally strictly properly $\bigvee$-definably isomorphic. By theorem 5.20, they can be uniquely extended to strictly properly $\bigvee$-definable homomorphisms $\overline{f} : X \longrightarrow Y$ and $\overline{g} : Y \longrightarrow X$. Both $\overline{g} \circ \overline{f} : X \longrightarrow X$ and 1_X are extensions of the inclusion $U \longrightarrow X$. By uniqueness of the extension, we have $\overline{g} \circ \overline{f} = 1_X$. Similarly, $\overline{f} \circ \overline{g} = 1_Y$. Thus X and Y are strictly properly $\bigvee$-definably isomorphic. The converse is clear. $\qquad\square$

Corollary 5.22 *Let X and Y be properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable groups and let $\widetilde{X}$ and $\widetilde{Y}$ be their universal strictly properly $\bigvee$-definable covering spaces. Then X and Y are locally strictly properly $\bigvee$-definably isomorphic iff $\widetilde{X}$ and $\widetilde{Y}$ are strictly properly $\bigvee$-definably isomorphic.*

5.6   The m-torsion points of a definable abelian group

In this subsection we describe $\pi_1(X)$ and the subgroup $X[m]$ of m-torsion points of a strictly properly $\bigvee$-definable abelian group X for which there is a definable group Z and a strictly properly $\bigvee$-definable covering space (X, p, Z).

Lemma 5.23 *Let X be properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable group for which there is a definable group Z and a strictly properly $\bigvee$-definable covering space (X, p, Z). Then $\pi_1(X)$ is a finitely generated abelian group.*

Proof. We have $\pi_1(X) \simeq p_*(\pi_1(X)) \leq \pi_1(Z)$ and by lemma 4.37, both $\pi_1(X)$ and $\pi_1(Z)$ are abelian groups. Therefore, its enough to show that $\pi_1(Z)$ is finitely generated. By [e1] there definable groups $V \trianglelefteq Z$ and $W \trianglelefteq U = Z/V$ such that V is the maximal definable, solvable normal subgroup of Z with no definably compact parts, W is the maximal definable, definably compact, abelian normal subgroup of U and U/W is a definably semisimple definable group. Moreover, $\pi_1(Z) \simeq \pi_1(V) \times \pi_1(W) \times \pi_1(U/W)$, $\pi_1(V) = 0$ and by [e2], $\pi_1(U/W)$ is a finite group. On the other hand, by theorem 3.15 $\pi_1(W) \simeq G(K, T)$ for some cell decomposition K of W with a maximal tree T and therefore, $\pi_1(W)$ is also finitely generated and the result follows. $\square$

Lemma 5.24 *Let X be properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable group and suppose that there is a definable group Z and a strictly properly $\bigvee$-definable covering space (X, p, Z). Then X has unbounded exponent, the subgroup $Tor(X)$ of torsion points of X is countable (in particular, if $\mathcal{N}$ is $\aleph_0$-saturated, then X has elements of infinite order) and, if X is properly $\bigvee$-definably complete and solvable then X is abelian.*

Proof. This follows from similar results for definable groups (see [e1] and [s]) together with theorem 5.15 and corollary 5.17. $\square$

Lemma 5.25 *Let X be a properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable abelian group. For $m \in \mathbb{N}$, let $m : X \longrightarrow X$ be the multiplication by m homomorphism. Then, $m_* : \pi_1(X) \longrightarrow \pi_1(X)$ is the homomorphism defined by $m_*([\Gamma]) = m[\Gamma]$.*

Proof. This is by induction on m. For $m = 1$ the result is clear, and if it is true for $m > 1$, then $(m+1)_*([\Gamma]) = [\Gamma(m \circ \Gamma)]$ (notation of lemma 4.37)$= [\Gamma \cdot (m \circ \Gamma)]$ (by lemma 4.37) $= [\Gamma][m \circ \Gamma] = (m+1)[\Gamma]$. $\square$

Theorem 5.26 *Let X be a properly $\bigvee$-definably connected, strictly properly $\bigvee$-definable abelian group. Suppose that there is a definable group Z and a strictly properly $\bigvee$-definable covering space (X, p, Z). Then X is divisible, $\pi_1(X)$ is a finitely generated torsion-free abelian group and for each $m \in \mathbb{N}$, the subgroup $X[m]$ of m-torsion points of X is a finite group isomorphic to $\pi_1(X)/m\pi_1(X)$. In particular, X is definably simply connected iff X is torsion-free.*

Proof. For $m \in \mathbb{N}$, let $m : X \longrightarrow X$ be the multiplication by m homomorphism. By lemma 5.24, $dim(m^{-1}(0)) = 0$. Since mX is a strictly properly $\bigvee$-definable subgroup of X with $dimX = dim(mX)$, we have $mX = X$ and X is divisible. By theorem 5.18, m is a strictly properly $\bigvee$-definable covering map and $X[m] \simeq \pi_1(X)/m_*\pi_1(X)$. By lemma 5.25, $X[m] \simeq \pi_1(X)/m\pi_1(X)$ and so, X is definably simply connected iff X is torsion-free. Since $\pi_1(X)$ is a strictly properly $\bigvee$-definable subgroup of the universal strictly properly $\bigvee$-definable covering $\widetilde{X}$ of X, which is torsion-free by the above, $\pi_1(X)$ is torsion-free. By lemma 5.23, $\pi_1(X)$ is finitely generated and the result follows. $\square$

In particular, assuming (as we conjecture), that for a definably compact, definably connected, definable abelian group X, $\pi_1(X) \simeq \mathbb{Z}^{dimX}$ it follows from theorem 5.26, that for every definably compact, definably connected, definable abelian group X, $X[m] \simeq (\mathbb{Z}/m\mathbb{Z})^{dimX}$ for every $m \in \mathbb{N}$.

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